

Modeling the complete ($\ell = 2, m = 0$) mode in aligned-spin and precessing binary black holes: applications to parameter estimation

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Background from: S. Ossokine, A. Buonanno (Max Planck Institute for Gravitational Physics),
Simulating eXtreme Spacetimes project, D. Steinhauser (Airborne Hydro Mapping GmbH)



Outline of the aligned-spin waveform model

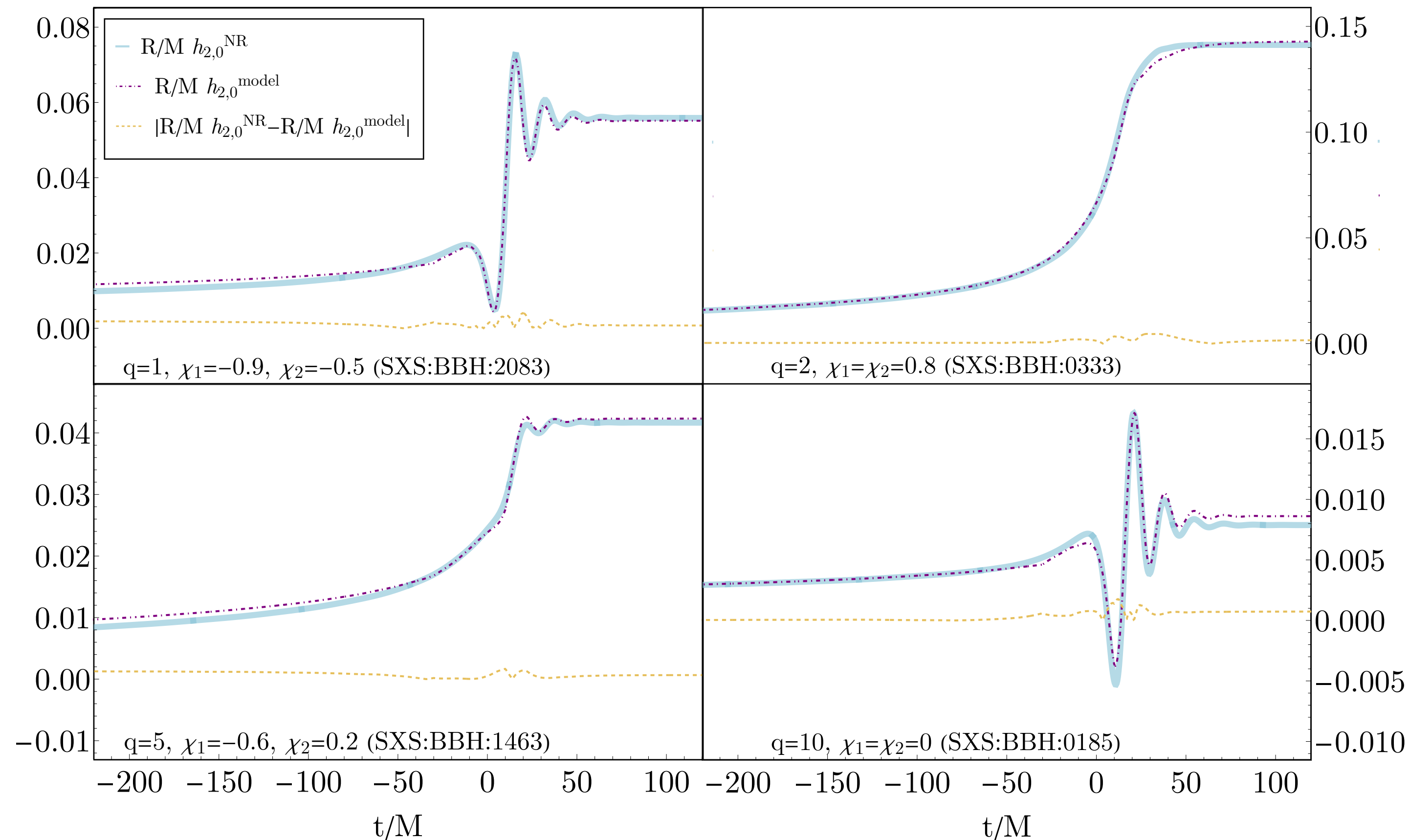
[[Rosselló-Sastre+\(2024\)](#)]



Outline of the aligned-spin waveform model [\[Rosselló-Sastre+\(2024\)\]](#)

Composed of two signals: QNM oscillations from the ringdown + displacement memory effect.

Phenomenological waveform model for the full (2,0) mode in time domain, for BBH mergers in quasi-circular orbits, with aligned spins. Parameter space: $q \leq 10, \chi_1, \chi_2 \in [-1, 1]$.



Implemented in IMRPhenomTHM in the [LAL infrastructure \[Estellés+\(2022\)\]](#) and in the
[Python package phenomxpy \[García-Quirós+\(2025\)\]](#).

Parameter estimation injections

[[Rosselló-Sastre+\(2025\)](#)]



Parameter estimation injections [\[Rosselló-Sastre+\(2025\)\]](#)

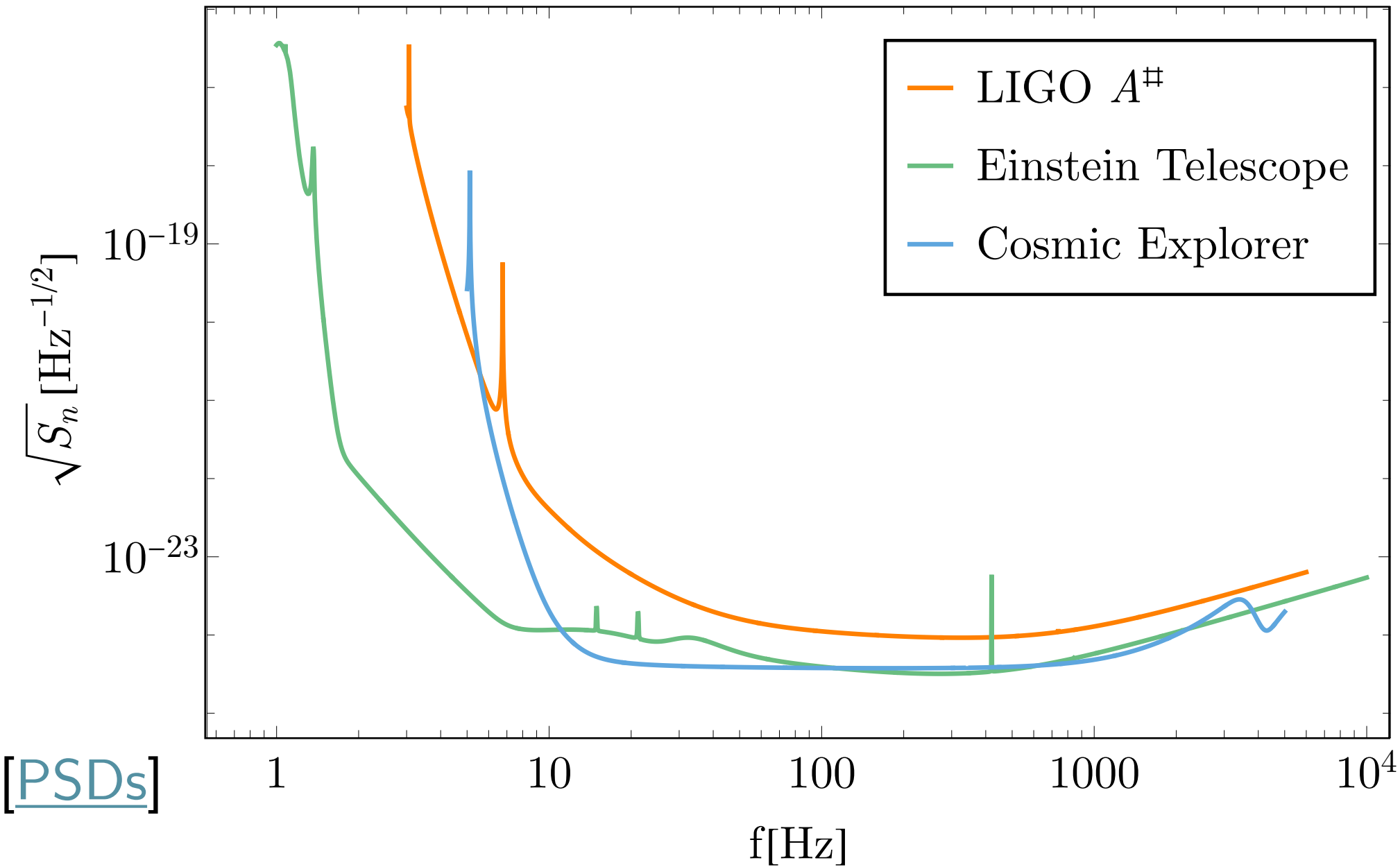
Setup of the zero-noise injections

- Previous results with **only** the **memory** in the (2,0) mode: [\[Xu+\(2024\)\]](#).
- Current results with the **full mode**: oscillatory + memory.
- Inject: IMRPhenomTHM+(2,0). Recover with IMRPhenomTHM and IMRPhenomTHM+(2,0) - LALSuite implementation.

Variable	Unit	Prior	Minimum	Maximum
$m_{1,2}^*$	$M_\odot$	Constraint	10	100
q	-	Uniform in Components	0.125	1
$\mathcal{M}^*$	$M_\odot$	Uniform in Components	15	100
$\chi_{1z,2z}$	-	Aligned Spin	-0.99	0.99
d_L	Mpc	Uniform Source Frame	20	1000
θ_{JN}	rad	Sin	0	π
ϕ	rad	Uniform	0	2π
δ	rad	Cos	$-\pi/2$	$\pi/2$
α	rad	Uniform	0	2π
ψ	rad	Uniform	0	π

* More restricted for the L1+H1+3G networks.

- bilby_pipe with dynesty sampler.
- nlive=500, naccept=15.
- gw-foutstep method for the Fourier Transform [\[Valencia+\(2024\)\]](#).
- 128 CPUs x 1 core.



- 8 injections in the L1+H1 ($A^\#$) network.
- 1 injection in the L1+CE network.
- 2 injections in the ET+L1+H1 network.

Parameter estimation injections [\[Rosselló-Sastre+\(2025\)\]](#)

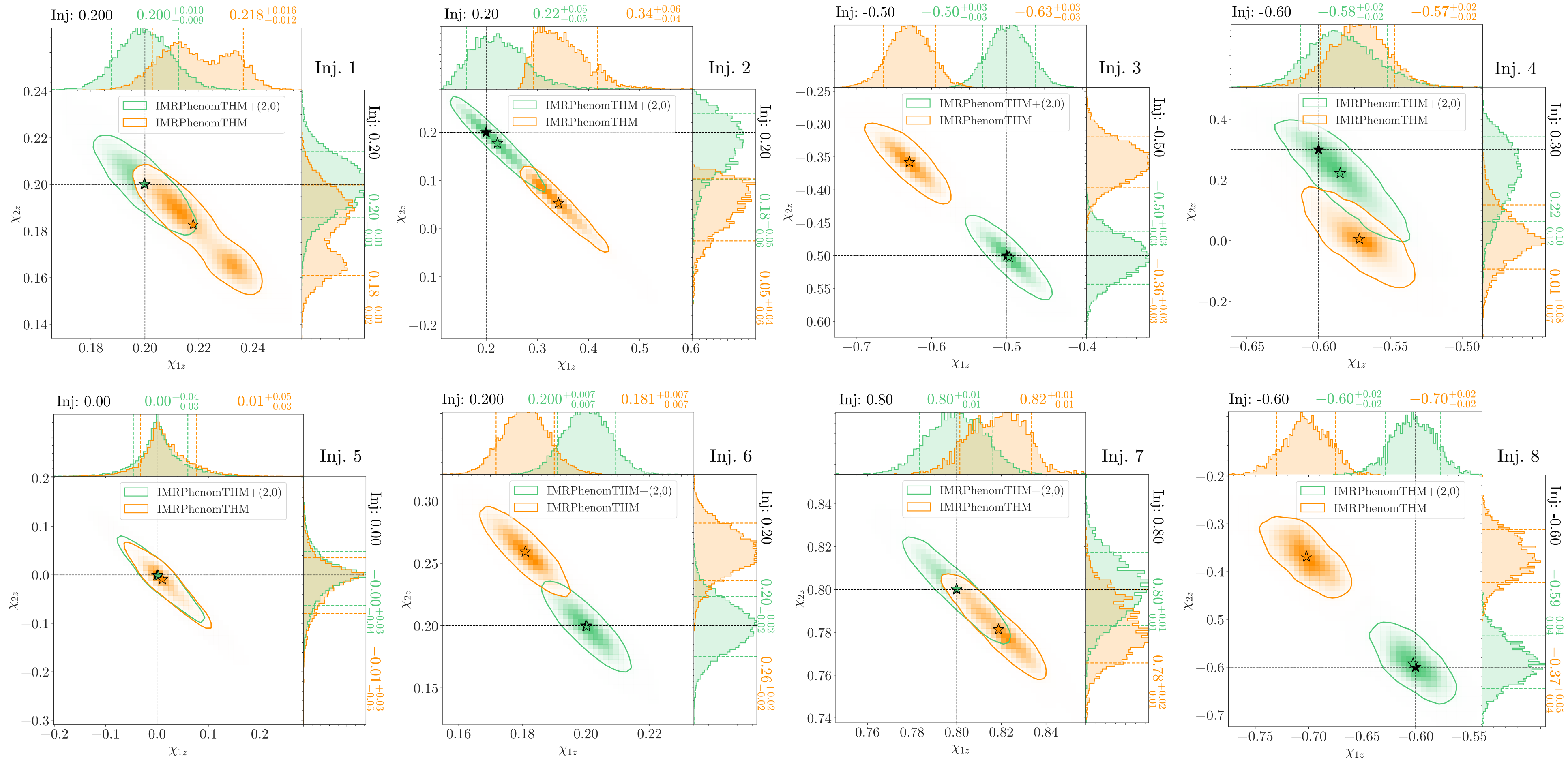
L1+H1 network

- $f_{\min} = f_{\text{ref}} = 10$ Hz, $f_{\text{sampling}} = 4096$ Hz, likelihood integration at 20 Hz.

Spin parameters

	$m_1[M_\odot]$	$m_2[M_\odot]$	q	$\mathcal{M}[M_\odot]$	χ_{1z}	χ_{2z}	$d_L[\text{Mpc}]$	$\theta_{\text{JN}}[\text{rad}]$
Injection 1	50.8	49.2	0.970	43.5	0.200	0.200	200	$\pi/2$
Injection 2	50.8	49.2	0.970	43.5	0.200	0.200	200	0
Injection 3	35.5	34.5	0.970	30.5	-0.500	-0.500	300	$\pi/2$
Injection 4	73.2	18.3	0.250	30.5	-0.600	0.300	300	$\pi/2$
Injection 5	20.3	19.7	0.970	17.4	0.000	0.000	400	$\pi/2$
Injection 6	71.5	35.8	0.500	43.5	0.200	0.200	200	$\pi/2$
Injection 7	50.8	49.2	0.970	43.5	0.800	0.800	300	$\pi/2$
Injection 8	50.1	25.1	0.500	30.5	-0.600	-0.600	300	$\pi/2$

$\delta = -0.492$ rad, $\alpha = 2.77$ rad, $\psi = 2.71$ rad, $\phi = 5.43$ rad, GW150914 trigger time.



Parameter estimation injections [\[Rosselló-Sastre+\(2025\)\]](#)

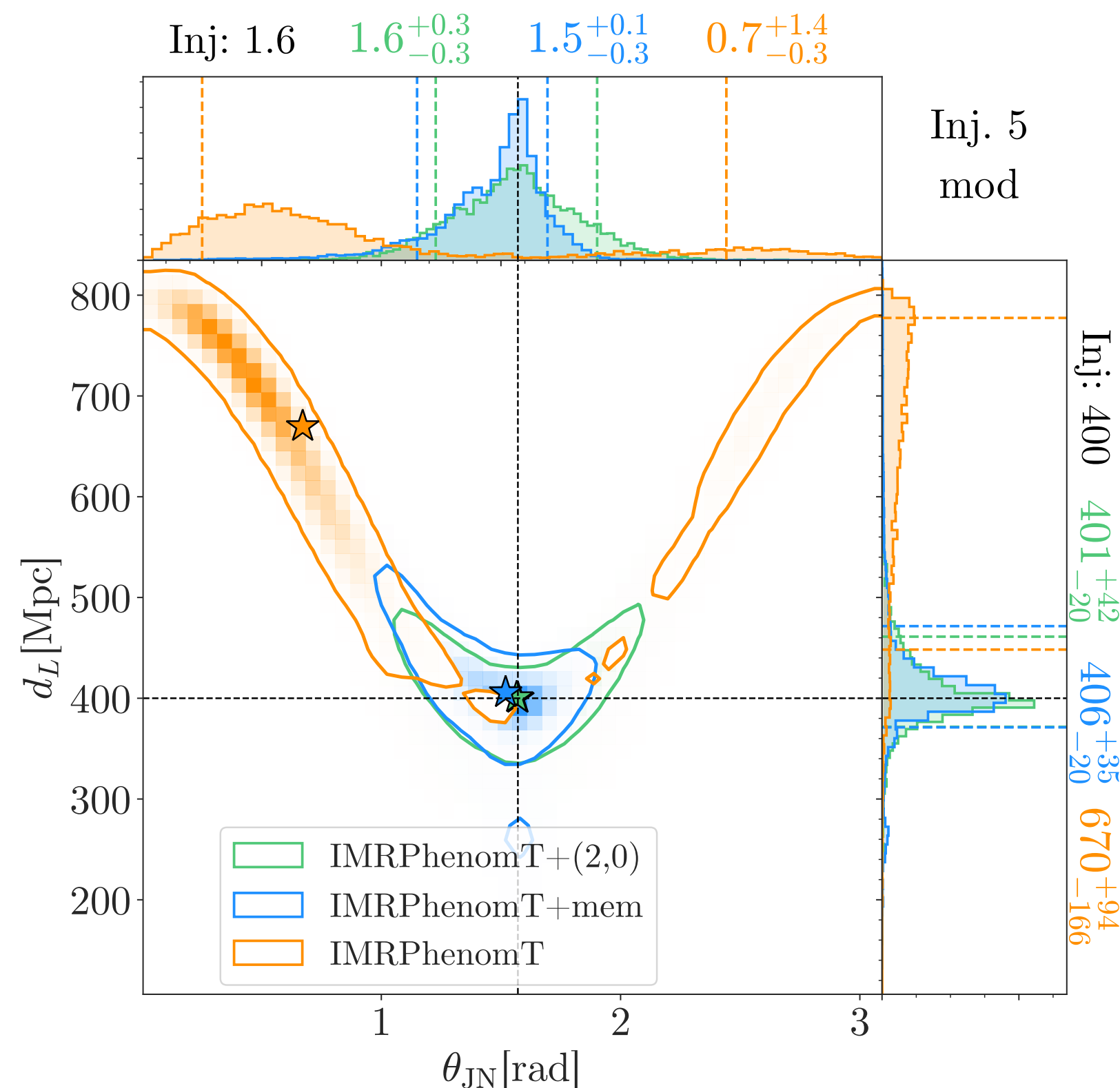
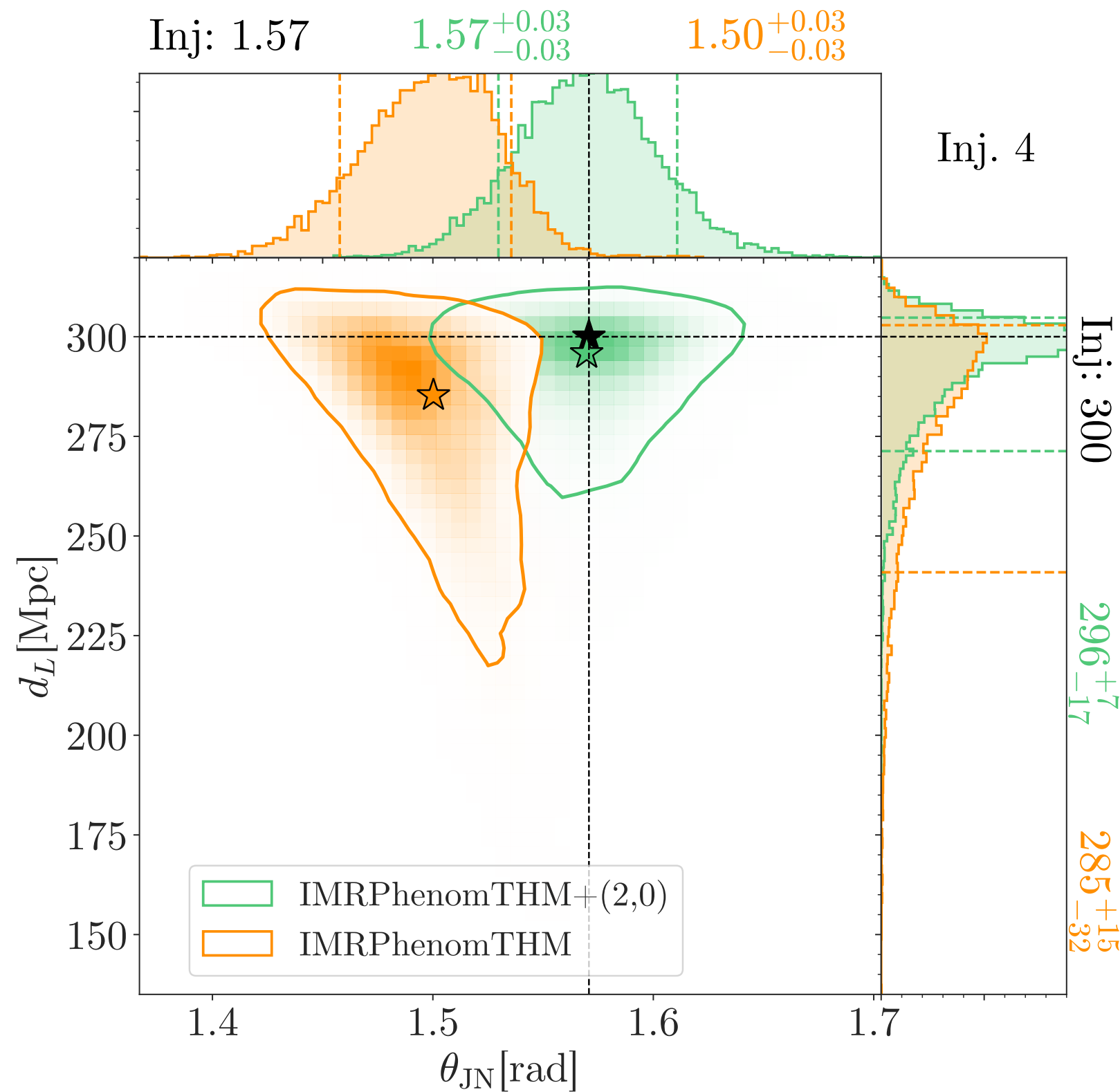
L1+H1 network

- $f_{\min} = f_{\text{ref}} = 10$ Hz, $f_{\text{sampling}} = 4096$ Hz, likelihood integration at 20 Hz.

	$m_1[M_\odot]$	$m_2[M_\odot]$	q	$\mathcal{M}[M_\odot]$	χ_{1z}	χ_{2z}	$d_L[\text{Mpc}]$	$\theta_{\text{JN}}[\text{rad}]$
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Distance-inclination degeneracy

$\delta = -0.492$ rad, $\alpha = 2.77$ rad, $\psi = 2.71$ rad, $\phi = 5.43$ rad, GW150914 trigger time.



$$-^2Y_{2,\pm 2}(\theta, \phi) = \sqrt{\frac{5}{64\pi}}(1 \pm \cos \theta)^2 e^{\pm 2i\phi}$$

$$-^2Y_{2,0}(\theta) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2 \theta$$

Comparison with [\[Xu+\(2024\)\]](#).

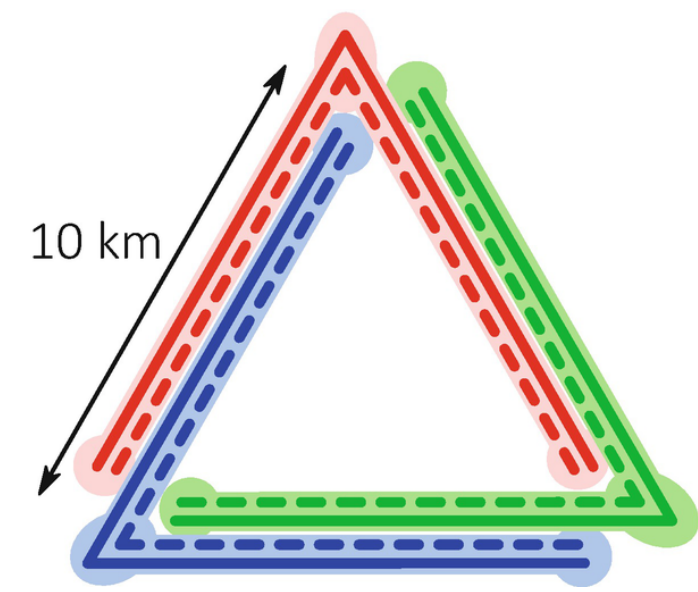
Parameter estimation injections [\[Rosselló-Sastre+\(2025\)\]](#)

ET+L1+H1 network

- $f_{\min} = 5$ Hz, $f_{\text{ref}} = 10$ Hz, $f_{\text{sampling}} = 4096$ Hz, likelihood integration at 10 Hz.

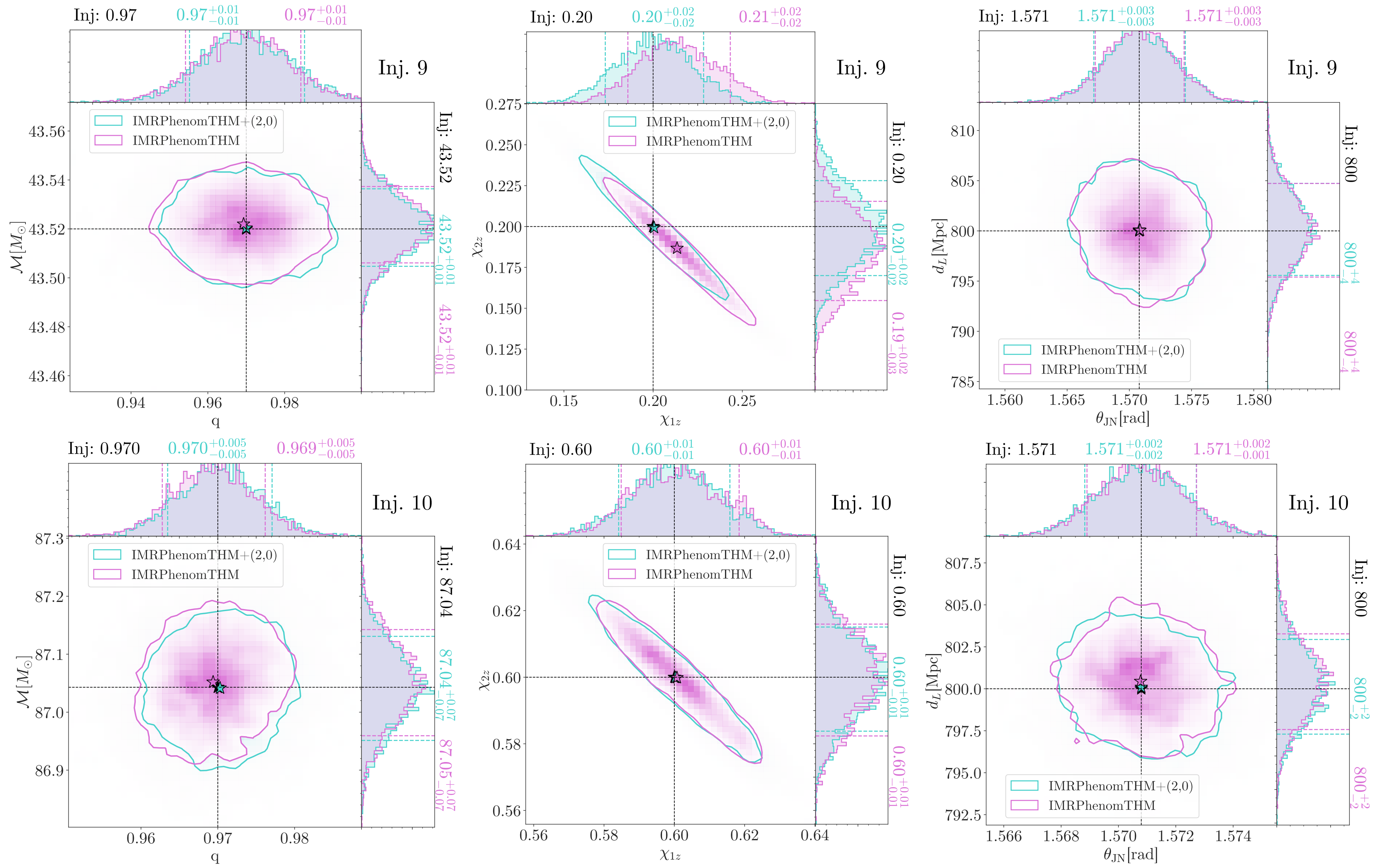
	$m_1[M_\odot]$	$m_2[M_\odot]$	q	$\mathcal{M}[M_\odot]$	χ_{1z}	χ_{2z}	$d_L[\text{Mpc}]$	$\theta_{\text{JN}}[\text{rad}]$
Injection 9	50.8	49.2	0.970	43.5	0.200	0.200	800	$\pi/2$
Injection 10	102	98.5	0.970	87.0	0.600	0.600	800	$\pi/2$

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ET triangular
configuration

[\[Lück+\(2022\)\]](#)



The precessing waveform model

[[Rosselló-Sastre+\(2025\)](#)]



The precessing waveform model [\[Rosselló-Sastre+\(2025\)\]](#)

Twisting up the oscillatory component

Twisting-up of the co-precessing modes using the
Wigner D -matrices:

$$h_{\ell,m}^{\text{I}} = \sum_{m'=-\ell}^{\ell} \mathcal{D}_{m,m'}^{\ell}(\alpha, \beta, \gamma) h_{\ell,m'}^{\text{cp}}(t)$$

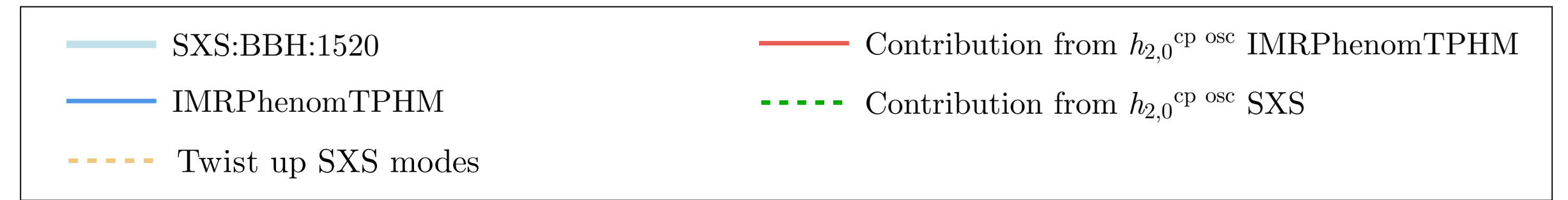
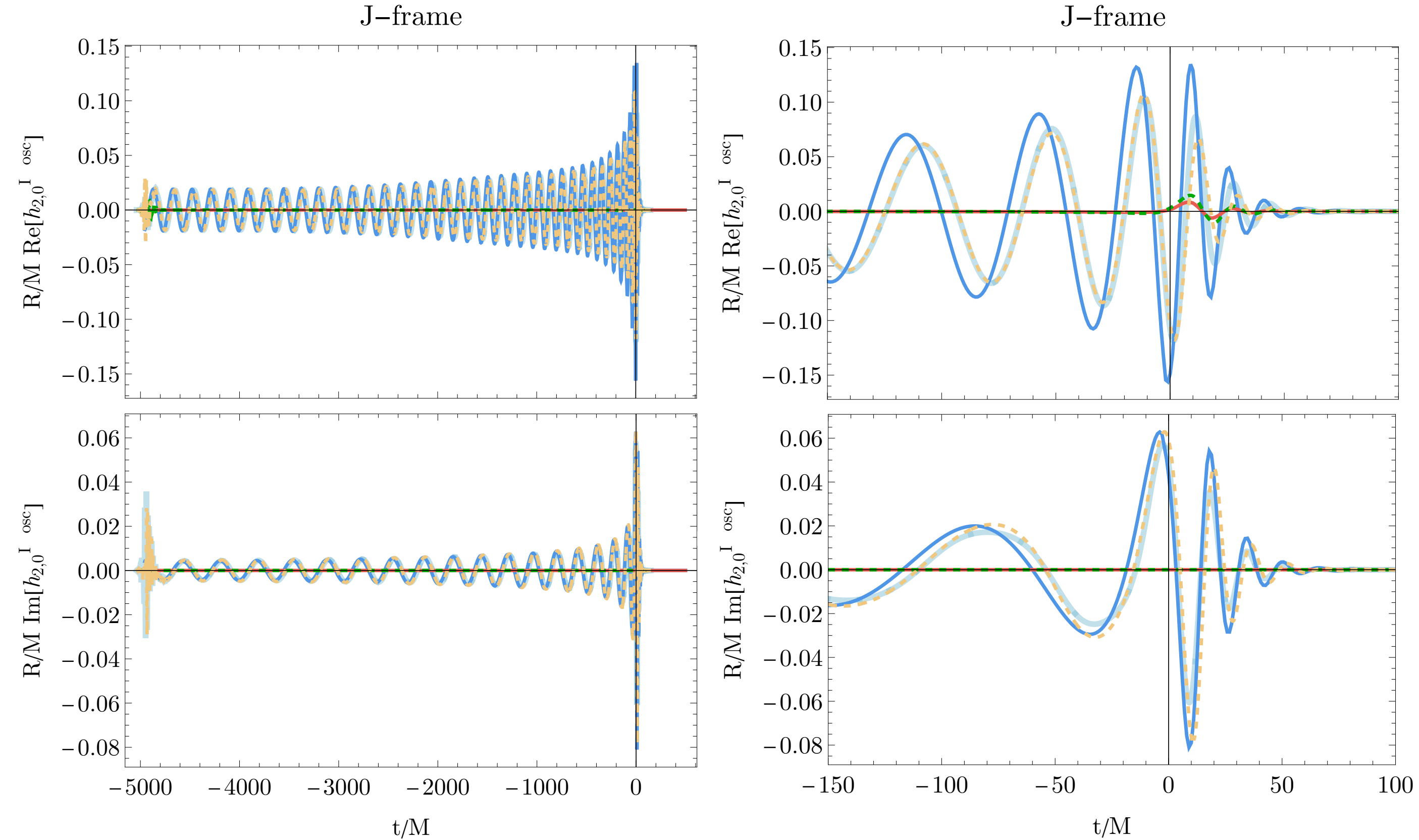
With the Euler angles:

$$\alpha = \arctan(\hat{L}_y / \hat{L}_x),$$

$$\cos \beta = \hat{\mathbf{z}} \cdot \hat{\mathbf{L}} = \hat{L}_z,$$

$$\dot{\gamma} = -\dot{\alpha} \cos \beta.$$

$$q = 3.03, \chi_1^{\text{ref}} = \{0.540, -0.137, -0.435\}, \chi_2^{\text{ref}} = \{0.056, 0.258, 0.129\}, \chi_{\text{p}} = 0.557.$$



The “twisted” (2,0) mode:

$$h_{2,0}^{\text{I osc}} = \frac{1}{4} \left\{ \boxed{h_{2,0}^{\text{cp osc}} [1 + 3 \cos(2\beta)]} + 2\sqrt{6} \left[\sin^2(\beta) \left(\text{Re} [h_{2,2}^{\text{cp}}] \cos(2\gamma) + \text{Im} [h_{2,2}^{\text{cp}}] \sin(2\gamma) \right) - i \sin(2\beta) \left(\text{Re} [h_{2,1}^{\text{cp}}] \sin(\gamma) - \text{Im} [h_{2,1}^{\text{cp}}] \cos(\gamma) \right) \right] \right\}$$

New term added

The precessing waveform model [\[Rosselló-Sastre+\(2025\)\]](#)

Twisting up the memory component

Memory integration from the BMS balance laws [\[Mitman+\(2021\)\]](#):

$$J_\varepsilon = \frac{1}{2} \bar{\partial}^2 \mathfrak{D}^{-1} \left[\frac{1}{4} \int_{u_1}^u |\dot{h}|^2 du \right] + \alpha$$

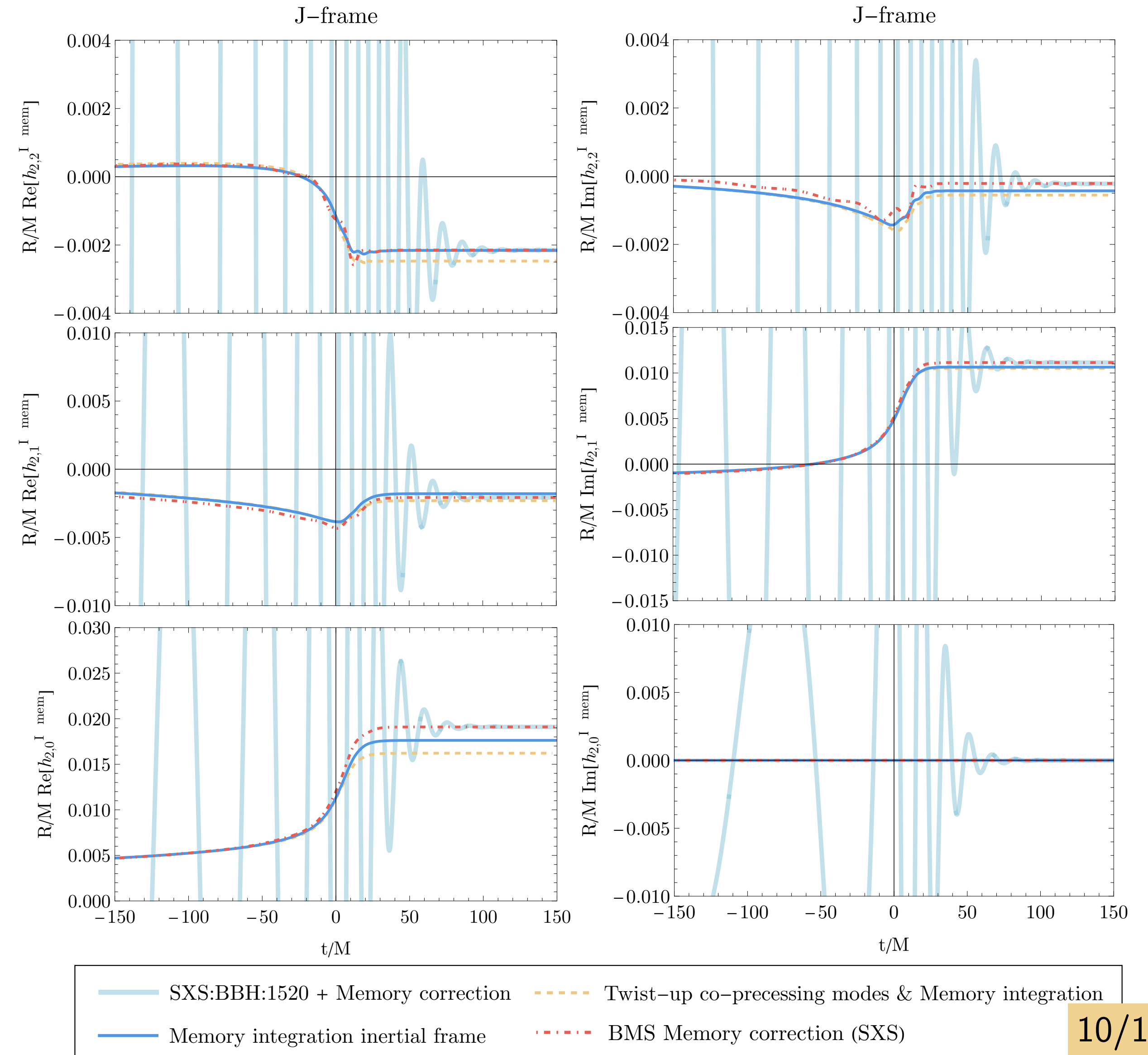
Twist-up the modes and compute the memory integral

$$h_{2,0}^{\text{I mem}} = \frac{1}{28} \sqrt{\frac{5}{6\pi}} \int_{u_1}^u \left\{ (1 + 3 \cos(2\beta)) \left[|\dot{h}_{2,2}^{\text{cp}}|^2 - \frac{1}{2} |\dot{h}_{2,1}^{\text{cp}}|^2 \right] \right. \\ \left. + \cos(2\gamma) \sin^2(\beta) \left[3 \left(\left(\text{Re} [\dot{h}_{2,1}^{\text{cp}}] \right)^2 - \left(\text{Im} [\dot{h}_{2,1}^{\text{cp}}] \right)^2 \right) + \sqrt{6} \text{Re} [\dot{h}_{2,2}^{\text{cp}}] \dot{h}_{2,0}^{\text{cp}} \right] \right. \\ \left. + \sin(2\gamma) \sin^2(\beta) \left[6 \text{Re} [\dot{h}_{2,1}^{\text{cp}}] \text{Im} [\dot{h}_{2,1}^{\text{cp}}] + 2\sqrt{6} \text{Im} [\dot{h}_{2,2}^{\text{cp}}] \dot{h}_{2,0}^{\text{cp}} \right] \right\} du$$

Memory integration of the inertial modes

$$h_{2,0}^{\text{I mem}} = -\frac{1}{14} \sqrt{\frac{5}{6\pi}} \int_{u_1}^u |\dot{h}_{2,0}^{\text{I}}|^2 du \\ + \frac{1}{14} \sqrt{\frac{5}{6\pi}} \int_{u_1}^u |\dot{h}_{2,2}^{\text{I}}|^2 du + \frac{1}{14} \sqrt{\frac{5}{6\pi}} \int_{u_1}^u |\dot{h}_{2,-2}^{\text{I}}|^2 du \\ - \frac{1}{28} \sqrt{\frac{5}{6\pi}} \int_{u_1}^u |\dot{h}_{2,1}^{\text{I}}|^2 du - \frac{1}{28} \sqrt{\frac{5}{6\pi}} \int_{u_1}^u |\dot{h}_{2,-1}^{\text{I}}|^2 du$$

$$q = 3.03, \chi_1^{\text{ref}} = \{0.540, -0.137, -0.435\}, \chi_2^{\text{ref}} = \{0.056, 0.258, 0.129\}, \chi_p = 0.557.$$



The precessing waveform model [\[Rosselló-Sastre+\(2025\)\]](#)

Two approaches that are not equivalent

1. Twisting up cp modes.
2. Compute memory integration.

$$h_{2,0}^{\text{I mem}} = \frac{1}{28} \sqrt{\frac{5}{6\pi}} \int_{u_1}^u (1 + 3 \cos(2\beta)) |\dot{h}_{2,2}^{\text{cp}}|^2 du,$$

$$h_{2,2}^{\text{I mem}} = \frac{1}{28} \sqrt{\frac{5}{\pi}} \int_{u_1}^u e^{-2i\alpha} \sin^2(\beta) |\dot{h}_{2,2}^{\text{cp}}|^2 du,$$

$$h_{2,-2}^{\text{I mem}} = \frac{1}{28} \sqrt{\frac{5}{\pi}} \int_{u_1}^u e^{2i\alpha} \sin^2(\beta) |\dot{h}_{2,2}^{\text{cp}}|^2 du,$$

$$h_{2,1}^{\text{I mem}} = -\frac{1}{14} \sqrt{\frac{5}{\pi}} \int_{u_1}^u e^{-i\alpha} \sin(\beta) \cos(\beta) |\dot{h}_{2,2}^{\text{cp}}|^2 du,$$

$$h_{2,-1}^{\text{I mem}} = \frac{1}{14} \sqrt{\frac{5}{\pi}} \int_{u_1}^u e^{i\alpha} \sin(\beta) \cos(\beta) |\dot{h}_{2,2}^{\text{cp}}|^2 du.$$



≠

1. Compute memory integration in cp frame.
2. Twist up memory contributions.

$$h_{2,0}^{\text{I mem}} = \frac{1}{4} (1 + 3 \cos(2\beta)) h_{2,0}^{\text{cp mem}},$$

$$h_{2,2}^{\text{I mem}} = \frac{1}{2} \sqrt{\frac{3}{2}} e^{-2i\alpha} \sin^2(\beta) h_{2,0}^{\text{cp mem}},$$

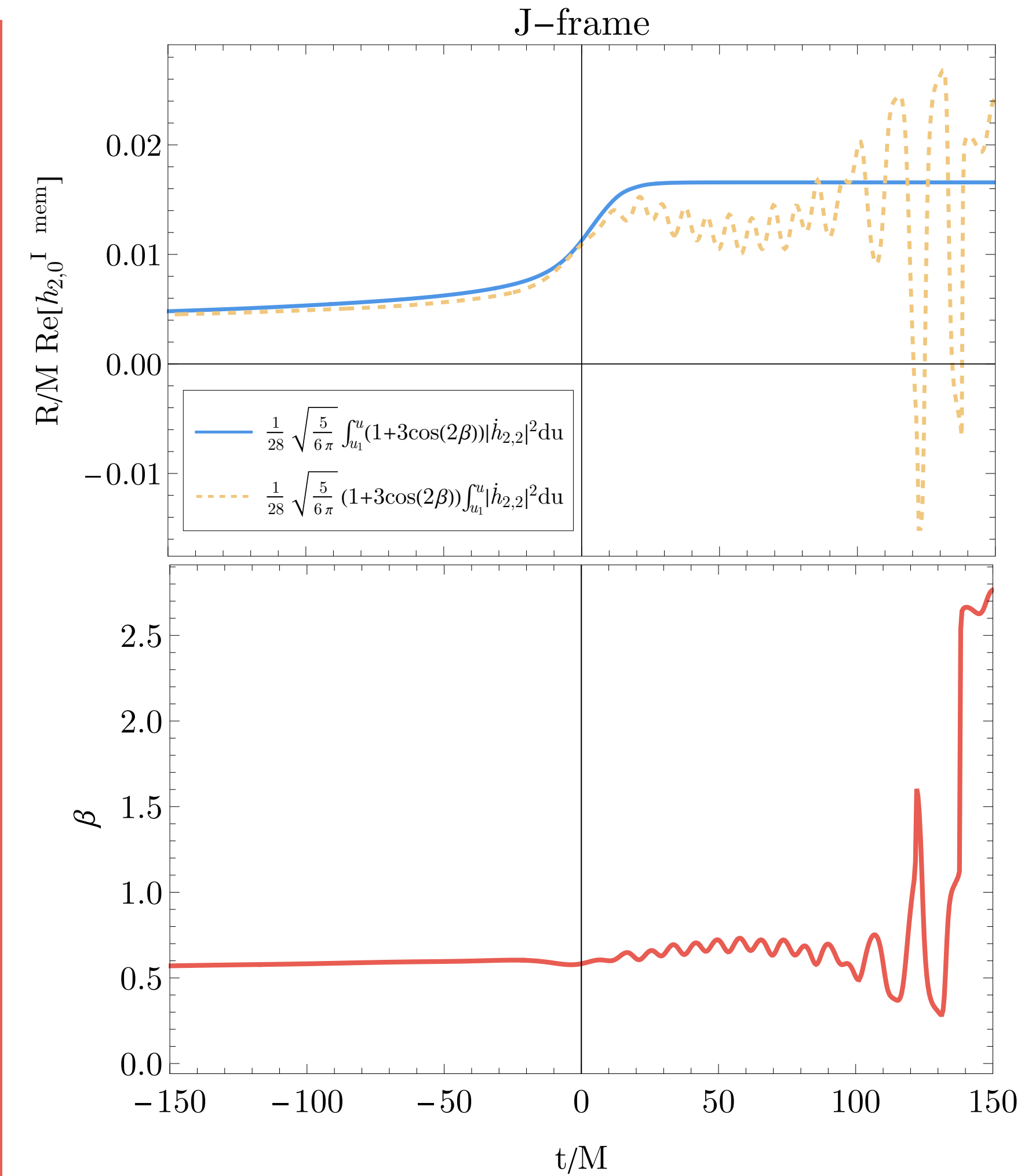
$$h_{2,-2}^{\text{I mem}} = \frac{1}{2} \sqrt{\frac{3}{2}} e^{2i\alpha} \sin^2(\beta) h_{2,0}^{\text{cp mem}},$$

$$h_{2,1}^{\text{I mem}} = -\sqrt{\frac{3}{2}} e^{-i\alpha} \sin(\beta) \cos(\beta) h_{2,0}^{\text{cp mem}},$$

$$h_{2,-1}^{\text{I mem}} = \sqrt{\frac{3}{2}} e^{i\alpha} \sin(\beta) \cos(\beta) h_{2,0}^{\text{cp mem}}.$$

With

$$\text{✗ } h_{2,0}^{\text{cp mem}} \approx h_{2,0}^{\text{AS mem}} = \frac{1}{7} \sqrt{\frac{5}{6\pi}} \int_{u_1}^u |\dot{h}_{2,2}^{\text{cp}}|^2$$



The precessing waveform model [\[Rosselló-Sastre+\(2025\)\]](#)

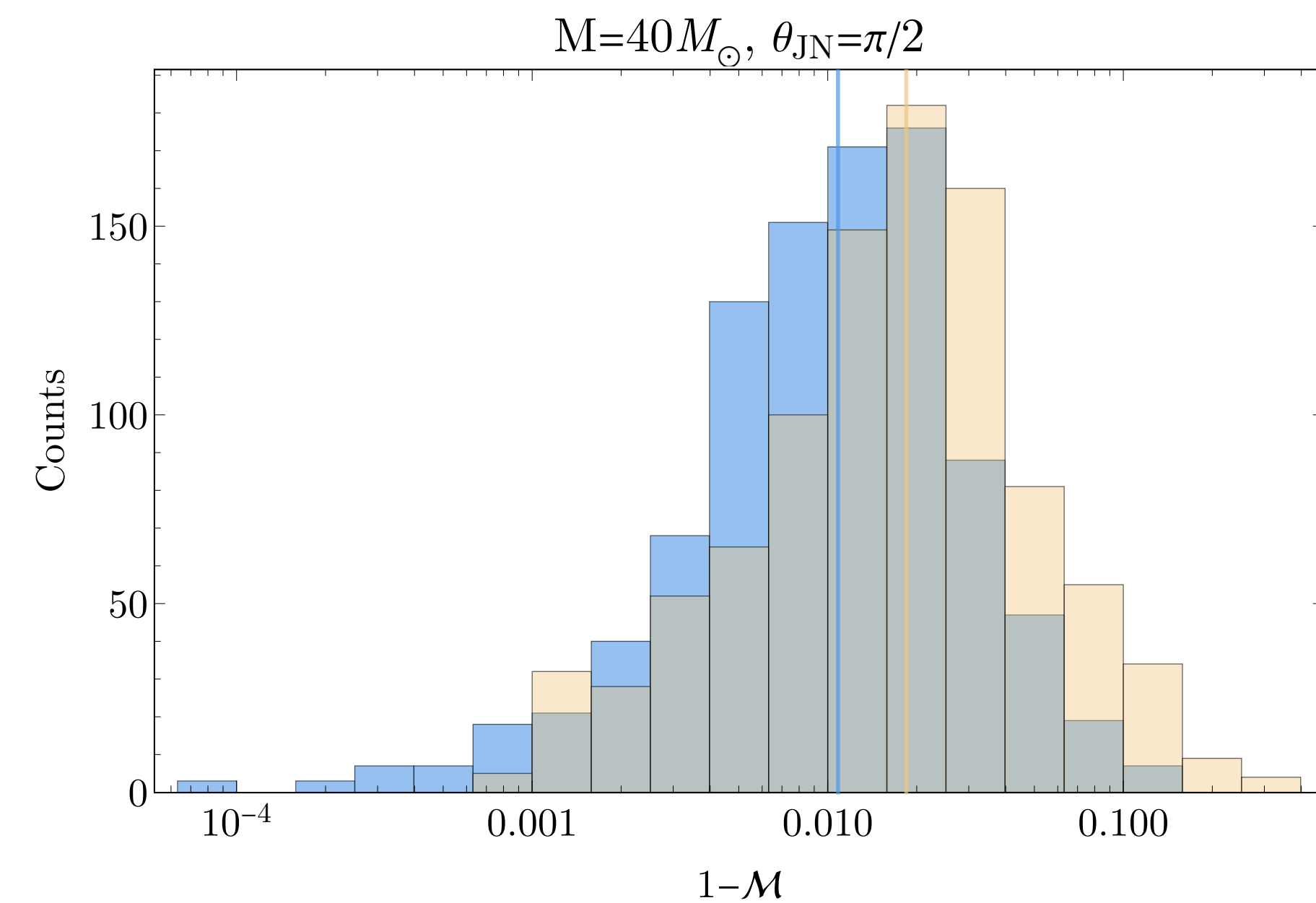
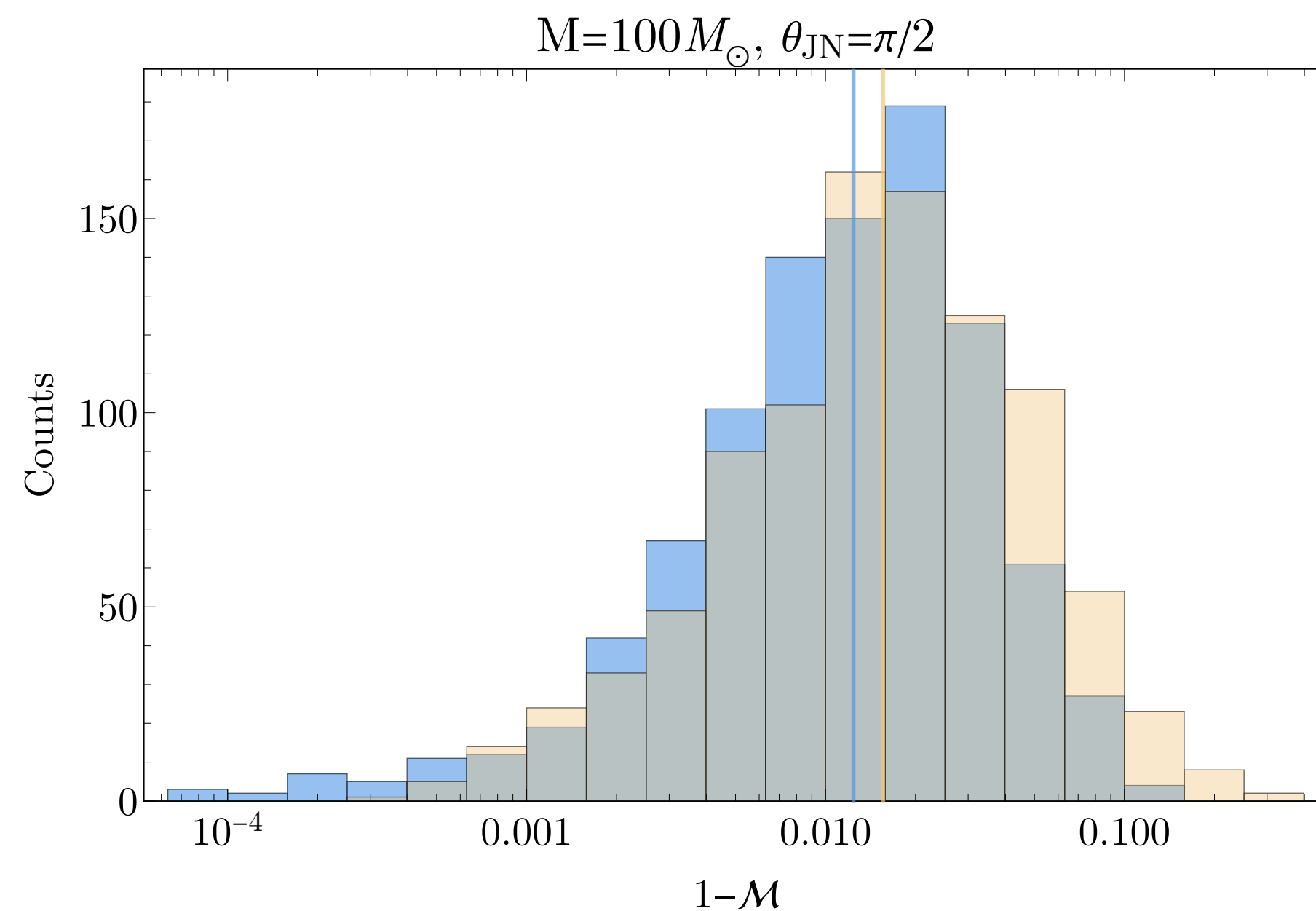
Model accuracy

Implementation within IMRPhenomTPHM in the [Python](#) package phenomxpy [\[García-Quirós+\(2025\)\]](#): flag to select the contributions included in the waveform.

[Mismatch calculation](#): model against 1076 SXS precessing simulations + memory.

$$1 - \mathcal{M} = 1 - \max_{\Delta t, \Delta \varphi} \frac{\langle h | g \rangle}{\sqrt{\langle h | h \rangle \langle g | g \rangle}}, \quad \langle h | g \rangle \equiv 4 \operatorname{Re} \left[\int_{f_{\min}}^{f_{\max}} \frac{\tilde{h}(f) \tilde{g}^*(f)}{S_n(f)} df \right]$$

[LIGO A# PSD](#)



■ SXS $\ell=2$ with memory vs TPHM $\ell=2$ with memory
■ SXS $\ell=2$ with memory vs TPHM $\ell=2$ no memory, no (2,0)

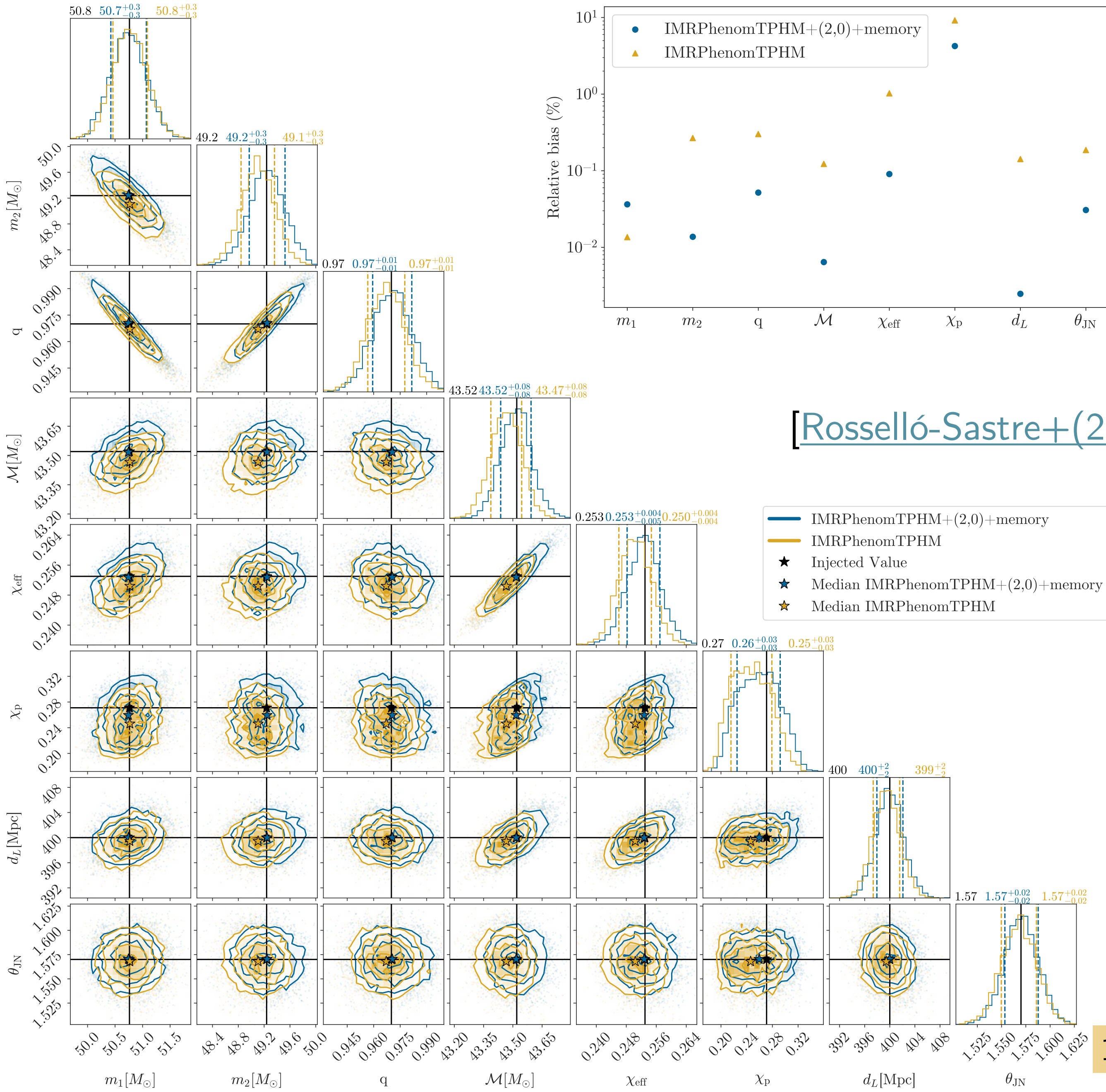
The precessing waveform model

Parameter estimation: zero-noise injection

- Injection: IMRPhenomTPHM+(2,0)+memory.
- Recovery:
 - IMRPhenomTPHM.
 - IMRPhenomTPHM+(2,0)+memory.

Variable	Unit	Prior	Minimum	Maximum
$m_{1,2}$	$M_{\odot}$	Uniform	5	100
q	-	Uniform in Components	0.125	1
$\mathcal{M}$	$M_{\odot}$	Uniform in Components	15	100
$a_{1,2}$	-	Uniform	0	0.99
θ_1	rad	Sin	0	π
θ_2	rad	Sin	0	π
ϕ_{12}	rad	Uniform	0	2π
ϕ_{JL}	rad	Uniform	0	2π
d_L	Mpc	Uniform Source Frame	20	1000
θ_{JN}	rad	Sin	0	π

- bilby_pipe with dynesty sampler.
- nlive=500, naccept=15.
- gw-foutstep method for the Fourier Transform [\[Valencia+\(2024\)\]](#).
- 112 CPUs x 1 core.
- LIGO A[#] sensitivity, SNR=235.



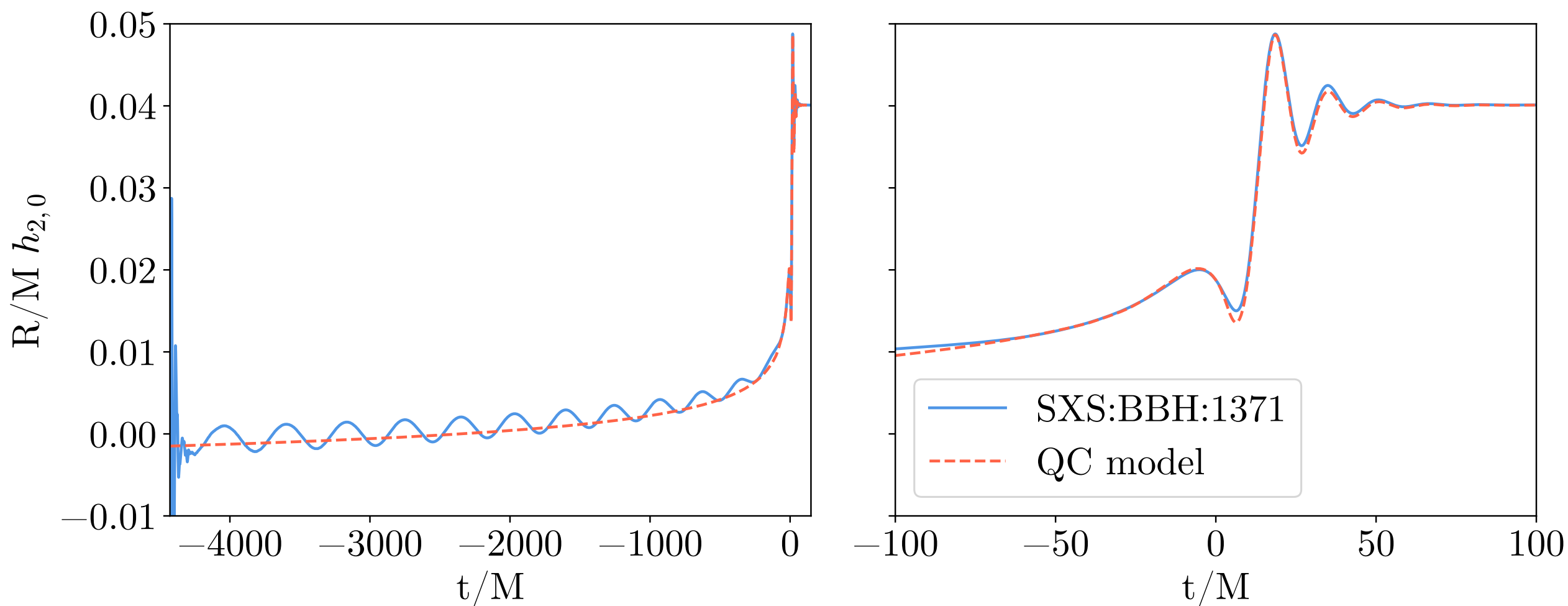
Preliminary results on eccentric (2,0)



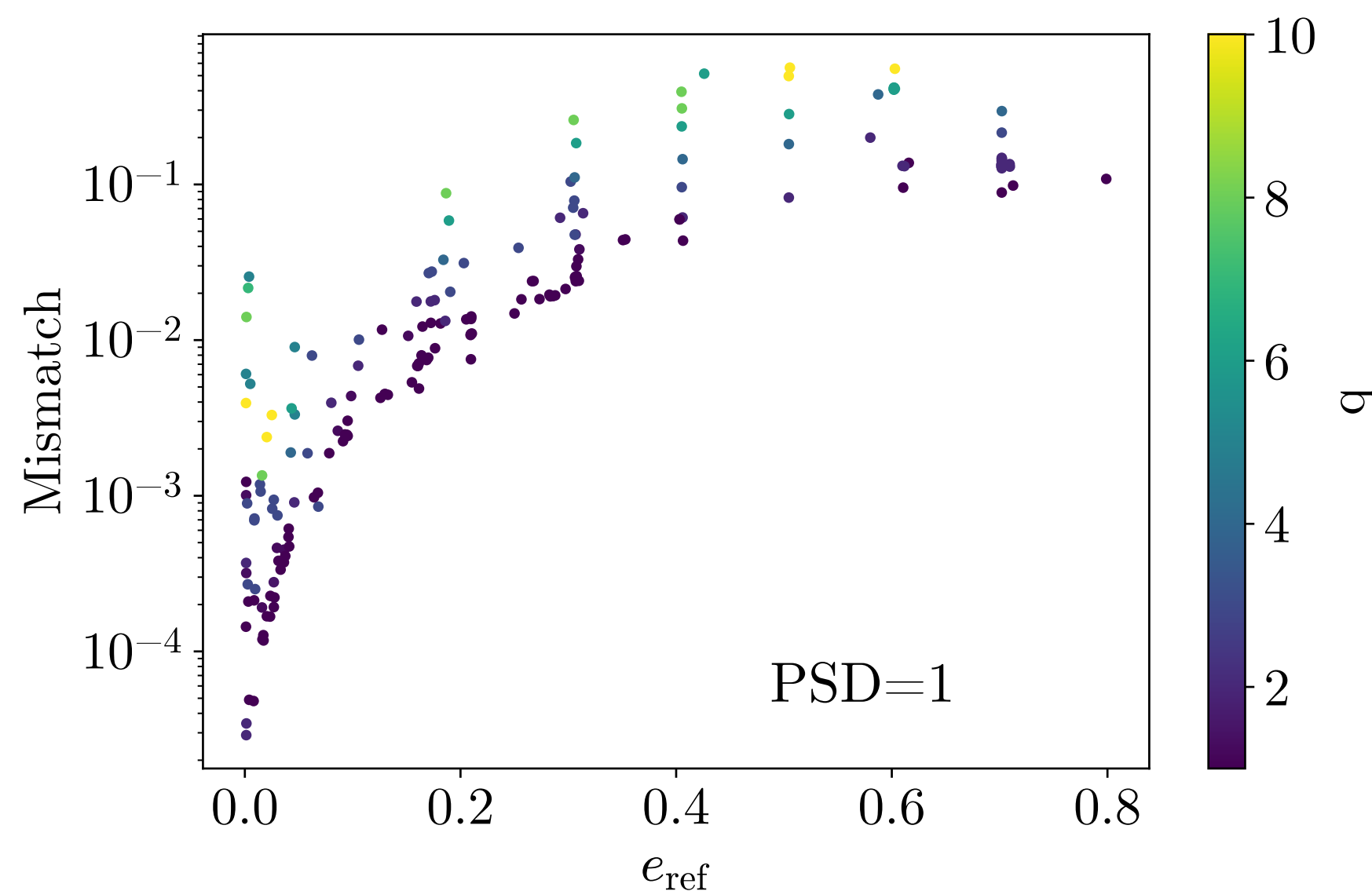
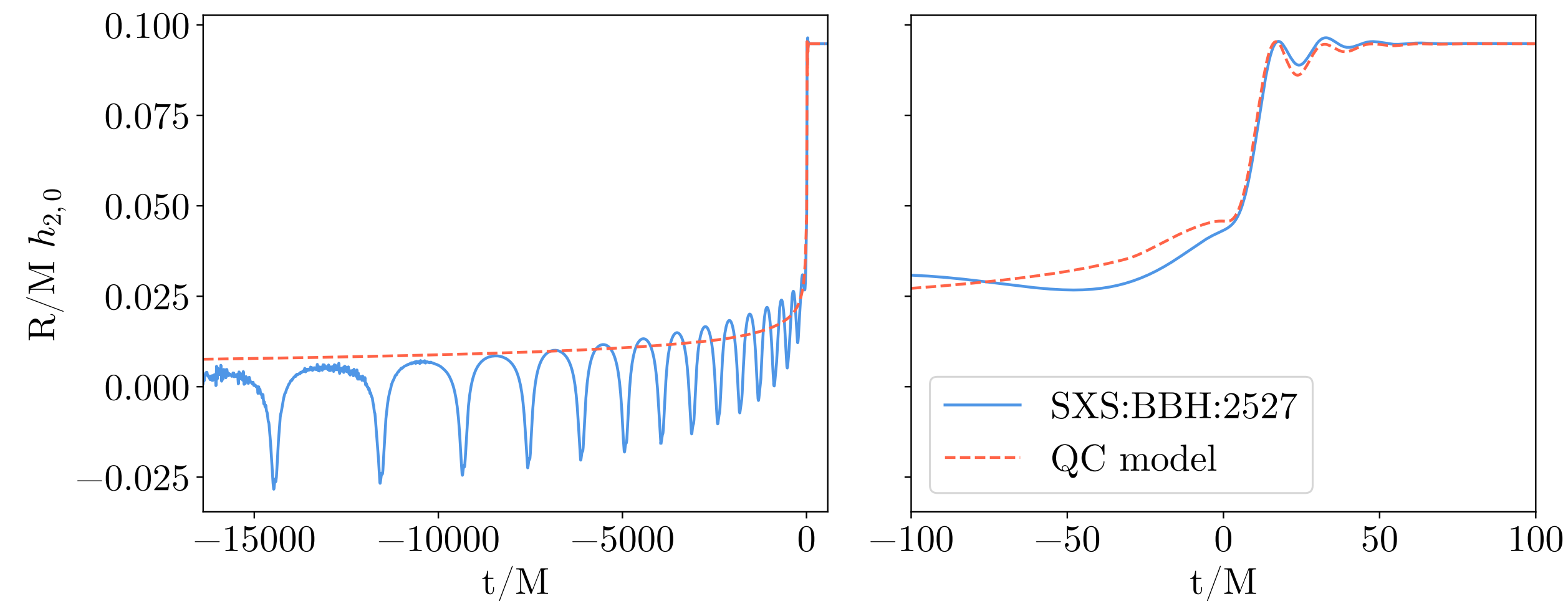
Preliminary results on eccentric (2,0)

First steps comparing the eccentric simulations from SXS with our quasi-circular model:

Low eccentricity $e_{\text{ref}}(f_{\text{ref}} = 0.017) = 0.11$

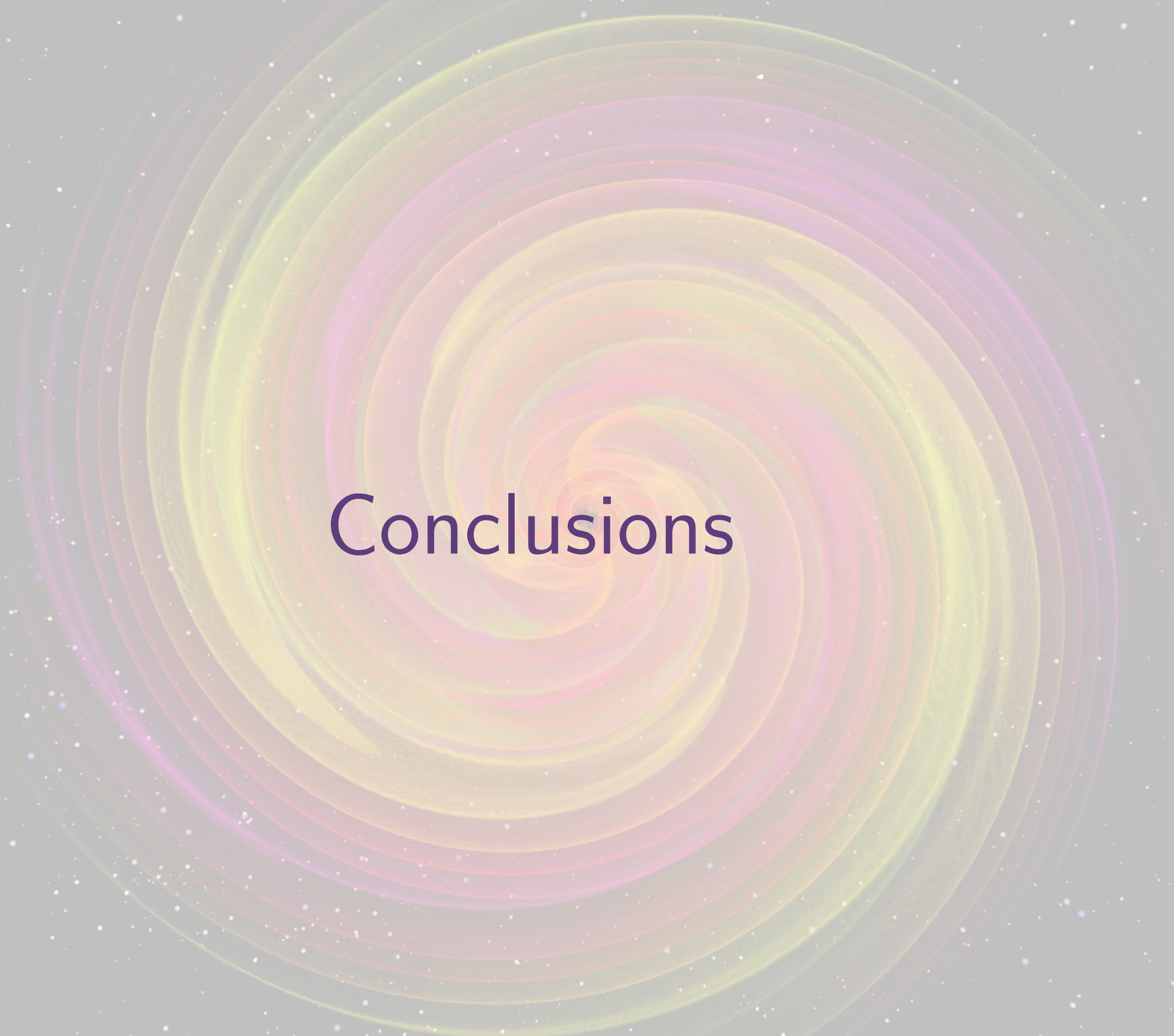


High eccentricity $e_{\text{ref}}(Mf_{\text{ref}} = 0.00045) = 0.80$



Future work: Implement eccentric corrections as in [IMRPhenomTEHM](#) [[Planas+\(2025\)](#)]:

3PN [oscillatory](#) and [DC memory](#) terms for non-spinning systems from [[Ebersold+\(2019\)](#), [Arun+\(2009\)](#)] and spinning systems from [[Henry+\(2024\)](#)].



Conclusions

Conclusions

- First Parameter Estimation injections using the full (2,0) mode.
- Biases in the individual spin components.
- Improvement in the distance-inclination measurement is negligible compared to other higher modes.
- More pronounced biases are expected at low masses, analysis is restricted by limited computational resources.
- Extended the AS waveform model for the full (2,0) mode to systems with spin precession.
- Used the “twisting-up” approximation for the oscillatory component.
- Developed analytical approaches for computing the displacement memory in all the $\ell = 2$ modes.
- Implementation in the phenomxpy Python package under the IMRPhenomTPHM waveform model, allowing independent selection of contributions.
- Performed a zero-noise injection test to evaluate biases from excluding the full (2,0) mode and memory effects.
- Future work: extension of the waveform model to eccentric systems.

Thank you!

Acknowledgments

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The authors thankfully acknowledge the computer resources at MareNostrum and Picasso and the technical support provided by Barcelona Supercomputing Center (BSC) through grants No. AECT-2024-3-0019 and AECT-2024-3-0027 from the Red Española Supercomputación (RES). The authors thank the Supercomputing and Bioinnovation Center (SCBI) of the University of Malaga for their provision of computational resources and technical support (www.scbi.uma.es/site).



Backup slides



BMS balance laws

Asymptotically flat spacetimes $\longrightarrow$ BMS symmetry group (infinite-dimensional generalization of the Poincaré group):

- Supertranslations.
- Superrotations.
- Superboosts.

Infinite set of symmetries $\longrightarrow$ Infinite set of balance laws

Related to soft hair and BH information paradox

Supermomentum balance law: $\delta^2 h = \int_{-\infty}^u |\dot{h}|^2 du - 4 \left(\Psi_2 + \frac{1}{4} \dot{h} \bar{h} \right) - 4M_{ADM}$ [Mitman+(2021)]

$$J \equiv \frac{1}{2} \bar{\delta}^2 \mathfrak{D}^{-1} \left[\frac{1}{4} \int_{-\infty}^u |\dot{h}|^2 du - \left(\Psi_2 + \frac{1}{4} \dot{h} \bar{h} \right) \right] + \alpha(\theta, \phi) \xrightarrow{\text{4 different contributions}} \left\{ \begin{array}{l} J = J_m + J_\varepsilon + J_{\hat{N}} + J_{\mathcal{J}} \\ J_m: \text{Bondi mass aspect.} \\ J_\varepsilon: \text{energy flux.} \\ J_{\hat{N}}: \text{angular momentum aspect.} \\ J_{\mathcal{J}}: \text{angular momentum flux.} \end{array} \right.$$

↑
unknown constant

Calculation of the memory only from the contribution of the energy flux:

$$J_\varepsilon = \frac{1}{2} \bar{\delta}^2 \mathfrak{D}^{-1} \left[\frac{1}{4} \int_{u_1}^u |\dot{h}|^2 du \right] + \alpha \left\{ \begin{array}{l} \text{Main contribution (AS case) into the (2,0) spherical harmonic:} \\ h_{2,0}^{\text{mem}} = \frac{1}{7} \sqrt{\frac{5}{6\pi}} \int_{u_1}^u |\dot{h}_{2,2}|^2 du - \frac{1}{14} \sqrt{\frac{5}{6\pi}} \int_{u_1}^u |\dot{h}_{2,1}|^2 du + \frac{5}{2\sqrt{42\pi}} \int_{u_1}^u \left(\dot{h}_{2,2}^{\text{Re}} \dot{h}_{3,2}^{\text{Re}} + \dot{h}_{2,2}^{\text{Im}} \dot{h}_{3,2}^{\text{Im}} \right) du - \frac{2}{11} \sqrt{\frac{2}{15\pi}} \int_{u_1}^u |\dot{h}_{4,4}|^2 du \end{array} \right.$$

Parameter estimation injections

L1+CE network

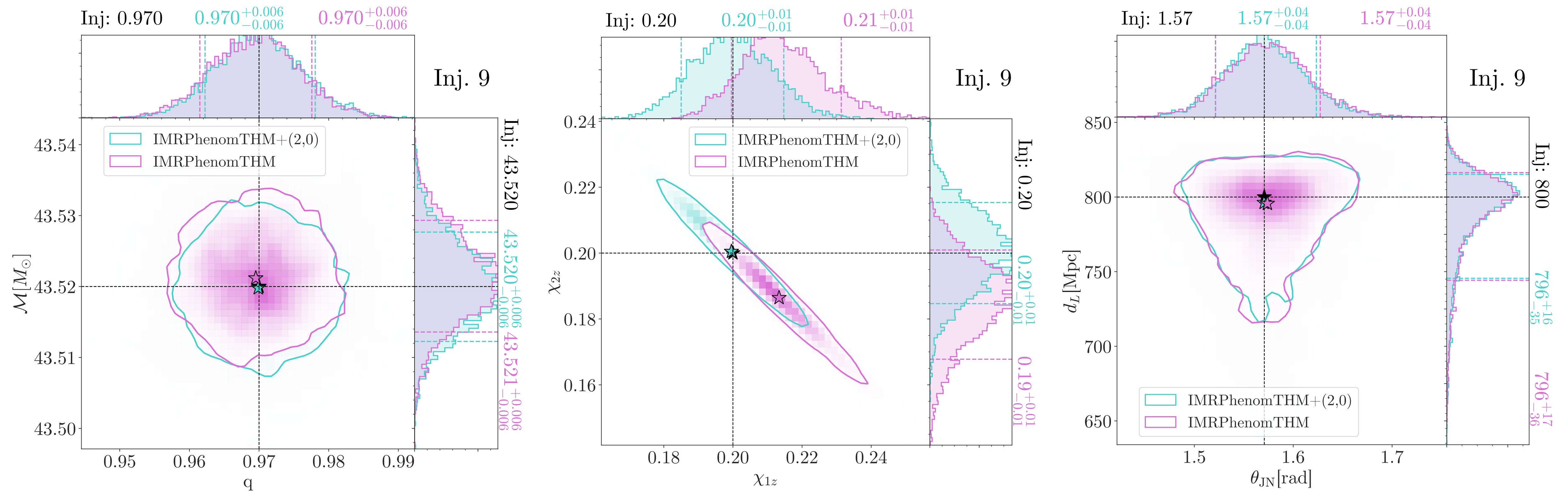
[Rosselló-Sastre+(2025)]

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$f_{\text{min}} = 5$ Hz, $f_{\text{ref}} = 10$ Hz, $f_{\text{sampling}} = 4096$ Hz, likelihood integration at 10 Hz.

CE: 40 km-long arms



SNR and runtimes of the injections

L1+H1 network

	Network SNR	Runtime	Runtime
	injected WF	IMRPhenomTHM+(2,0)	IMRPhenomTHM
Injection 1	456	14h 52min	14h 49min
Injection 2	930	13h 35min	12h 43min
Injection 3	192	17h 56min	17h 19min
Injection 4	184	17h 21min	17h 6min
Injection 5	107	1d 21h 18min	1d 17h 25min
Injection 6	435	15h 41min	15h 30min
Injection 7	370	15h 38min	15h 13min
Injection 8	174	21h 21min	18h 36min

L1+H1+3G networks

	Detector	Network SNR	Runtime	Runtime
	network	injected WF	IMRPhenomTHM+(2,0)	IMRPhenomTHM
Injection 9	CE+L1	594	3d 15h 33min	3d 12h 1min
Injection 9	ET+L1+H1	257	3d 6h 59min	3d 3h 7min
Injection 10	ET+L1+H1	484	23h 52min	23h 39min

Negligible terms in the full expression of $h_{2,0}^{\text{I mem}}$

$$\begin{aligned}
 h_{2,0}^{\text{I mem}} = & \frac{1}{224} \sqrt{\frac{5}{6\pi}} \int_{u_1}^u \left\{ 48 \sin^2(\beta) \sin(4\gamma) \left[\left(\text{Im} [h_{2,2}^{\text{cp}}] \sin(\beta) \dot{\alpha} - \text{Re} [h_{2,2}^{\text{cp}}] \dot{\beta} \right) \left(\text{Im} [h_{2,2}^{\text{cp}}] \dot{\beta} + \text{Re} [h_{2,2}^{\text{cp}}] \sin \beta \dot{\alpha} \right) \right] \right. \\
 & + 12 \sin^2(\beta) \cos(4\gamma) \left[2 \left((\text{Im} [h_{2,2}^{\text{cp}}])^2 - (\text{Re} [h_{2,2}^{\text{cp}}])^2 \right) \left(\dot{\beta}^2 - \sin^2(\beta) \dot{\alpha}^2 \right) + 8 \text{Im} [h_{2,2}^{\text{cp}}] \text{Re} [h_{2,2}^{\text{cp}}] \sin(\beta) \dot{\beta} \dot{\alpha} \right] \\
 & + 8 \text{Re} [\dot{h}_{2,2}^{\text{cp}}] \left[4 \text{Im} [h_{2,2}^{\text{cp}}] (1 + 3 \cos(2\beta)) \dot{\gamma} + \text{Im} [h_{2,2}^{\text{cp}}] (13 \cos(\beta) + 3 \cos(3\beta)) \dot{\alpha} - 6 \text{Re} [h_{2,2}^{\text{cp}}] \sin(2\beta) \dot{\beta} \right] \\
 & - 8 \text{Im} [\dot{h}_{2,2}^{\text{cp}}] \left[4 \text{Re} [h_{2,2}^{\text{cp}}] (1 + 3 \cos(2\beta)) \dot{\gamma} + \text{Re} [h_{2,2}^{\text{cp}}] (13 \cos(\beta) + 3 \cos(3\beta)) \dot{\alpha} + 6 \text{Im} [h_{2,2}^{\text{cp}}] \sin(2\beta) \dot{\beta} \right] \\
 & + |h_{2,2}^{\text{cp}}|^2 \left[16(13 \cos(\beta) + 3 \cos(3\beta)) \dot{\gamma} \dot{\alpha} - 4(1 + 3 \cos(2\beta)) \left(\dot{\beta}^2 - 8 \dot{\gamma}^2 \right) + (60 \cos(2\beta) + 3 \cos(4\beta) + 65) \dot{\alpha}^2 \right] \\
 & \left. + 8(1 + 3 \cos(2\beta)) \left| \dot{h}_{2,2}^{\text{cp}} \right|^2 \right\} du
 \end{aligned}$$

