Tidal effects in compact binaries within the post-Newtonian framework

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Based on several works

1. [HFB19] Q.H., G. Faye, L. Blanchet, arXiv:1912.01920
2. [HFB20a] Q.H., G. Faye, L. Blanchet, arXiv:2005.13367
3. [HFB20b] Q.H., G. Faye, L. Blanchet, arXiv:2009.12332
4. [DHB24] E. Dones, Q.H., L. Bernard, arXiv:2412.14249

Outline

1) Tidal effects

2) PN formalism

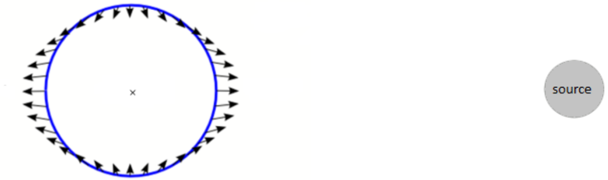
3) Effective action for adiabatic tides

4) Full waveform for quasi-circular orbits

5) Summary and perspectives

The basics of tidal effects

↪ The presence of an extended body in a non-homogeneous gravitational field induces a deformation.



↪ The deformability of the body is parametrized with **Love numbers**.

↪ We distinguish two types of tidal effects

- **Adiabatic (or static) tides**: the extended body is at thermodynamical equilibrium at each instant
- **Dynamical tides**: presence of vibration modes that can enter in resonance with the orbital motion

Motivations for tidal effects within the PN approach

- Compare with other analytical resolution methods (QFT, GSF...)
  $\hookrightarrow$ Better understanding of the theory

- Improve the precision of waveform models describing matter effects
  $\hookrightarrow$ PN results are used in (almost) all waveform models

	GW150914	GW170817
Chirp mass ($M_\odot$)	$30.4^{+2.1}_{-1.9}$	$1.188^{+0.004}_{-0.002}$
Cycles	8	~ 3000

  $\hookrightarrow$ NR inadequate for high number of cycles
  $\hookrightarrow$ Necessity to use analytical results in waveform models

- Constrain the equation of state of neutron stars

- Allows an independent measurement of the Hubble-Lemaître constant

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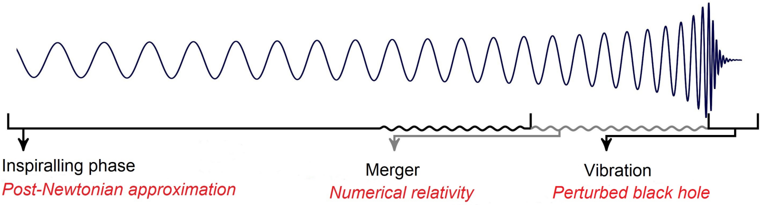
The post-Newtonian formalism



PN formalism:

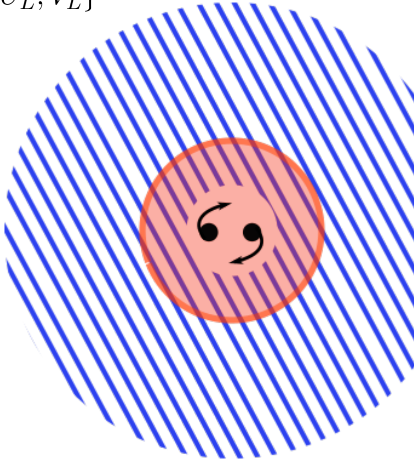
- Perturbative expansion of the equations of GR

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu} \quad \text{with} \quad \tau^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}(h, \partial h, \partial^2 h)$$

- Small velocities, weak field: $(v/c)^2 \sim GM/rc^2 \ll 1$
- Suitable to describe the inspiral of compact binaries
- nPN order $\rightarrow O(1/c^{2n})$ beyond leading order $\Rightarrow 2.5\text{PN} = O(1/c^5)$
- Can be applied to any matter action

The PN-MPM formalism in a nutshell

- Near zone: PN expansion ($v/c \ll 1$)
- Exterior zone: PM expansion ($G \ll 1$), source moments $\{I_L, J_L\}$
- Buffer zone: matching PN and PM multipole expansions
- Radiative zone: radiative moments $\{U_L, V_L\}$



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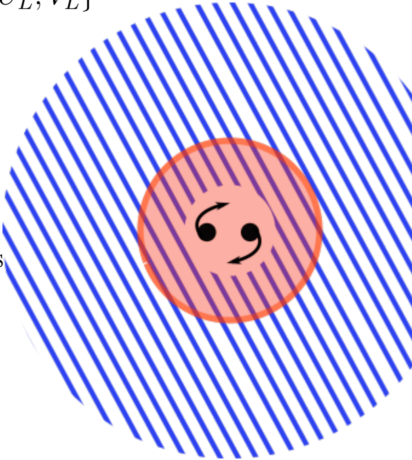

In practice, we perform two computations

- Conservative sector
  - $\hookrightarrow$ Conservative dynamics
  - $\hookrightarrow$ Conserved quantities
- Radiative sector
  - $\hookrightarrow$ Radiated fluxes
  - $\hookrightarrow$ Observables: phase and amplitude of the GW

Brief overview of the steps to compute the full waveform

$$h_+ - ih_\times = \sum_{\ell\geq 2}\sum_{|m|\leq\ell} {}_{-2}Y_{\ell m}\, h_{\ell m}(t)\, e^{-im\phi(t)}$$

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$$I_{ij} = \int \mathrm{d}^3 x \, \hat{x}^{ij} \left(\bar{\Sigma} + \dots \right)$$

$\bar{\Sigma}$ related to the PN metric and the stress-energy tensor

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Effective action at 2PN for adiabatic tides

$$S = S_{\text{EH}} + S_{\text{pp}} + S_{\text{T}}$$

$$\boxed{S_{\text{T}} = \sum_{\text{A}=1,2} \int \mathrm{d}\tau_A \left[\frac{\mu_A^{(2)}}{4} G^A_{\alpha\beta} G_A^{\alpha\beta} + \frac{\sigma_A^{(2)}}{6c^2} H^A_{\alpha\beta} H_A^{\alpha\beta} + \frac{\mu_A^{(3)}}{12} G^A_{\alpha\beta\gamma} G_A^{\alpha\beta\gamma} \right]}$$

- $G_{\alpha\beta} \equiv -R_{\alpha\mu\beta\nu} u^\mu u^\nu$: tidal quadrupole moment (mass type)
- $H_{\alpha\beta} \equiv 2R^*_{\alpha\mu\beta\nu} u^\mu u^\nu$: tidal quadrupole moment (current type)
- $G_{\alpha\beta\gamma}$: tidal octupole moment (mass type)

[Bini, Damour, Faye 2012]

Effective action at 2PN for adiabatic tides

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- $\mu^{(\ell)}$ and $\sigma^{(\ell)}$ linked to the Love numbers : $\mu_A^{(\ell)} \propto k_A^{(\ell)} R_A^{2\ell+1}$
- $\mathscr{C} = \dfrac{Gm}{Rc^2} \sim 1$ for compact objects
- $\mu^{(2)} \sim O\left(\dfrac{1}{c^{10}}\right) \sim \sigma^{(2)} \to$ Newtonian (leading) order (5PN)
- $\mu^{(3)} \sim O\left(\dfrac{1}{c^{14}}\right) \to$ 2PN relative (7PN)

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Intermediate computations

1) Conservative sector [HFB19, HFB20b]

- Compute the Fokker action and associated reduced Lagrangian
- Deduce the EOM and the 10 conserved quantities $\{E, J^i, P^i, G^i\}$

2) Radiative sector [HFB20a, DHB24]

- Compute the stress-energy tensor

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

- Compute the PN metric $g_{\mu\nu}$ sourced by $T^{\mu\nu}$
- Deduce the source multipole moments of the system
- Compute the non-linear effects to obtain the radiative multipoles
- Compute the energy and angular momentum fluxes
- Compute the phase and amplitude

Results: binding energy to 2PN for circular orbits

$$x \equiv \left(\frac{GM\omega}{c^3}\right)^{2/3} \qquad \nu \equiv \frac{m_1 m_2}{M^2} \qquad \delta \equiv \frac{m_1 - m_2}{M}$$

$$\begin{aligned}
E \;=\; & -\frac{Mc^2\nu x}{2}\Bigg\{1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right)x^2 \\
& -18\,\widetilde{\mu}_+^{(2)} x^5 + \left[\left(-\frac{121}{2} + 33\nu\right)\widetilde{\mu}_+^{(2)} - \frac{55}{2}\delta\,\widetilde{\mu}_-^{(2)} - 176\widetilde{\sigma}_+^{(2)}\right]x^6 \\
& + \left[\left(-\frac{20865}{56} + \frac{5434}{21}\nu - \frac{91}{4}\nu^2\right)\widetilde{\mu}_+^{(2)} + \delta\,\widetilde{\mu}_-^{(2)}\left(-\frac{11583}{56} + \frac{715}{12}\nu\right)\right. \\
& \left. + \left(-\frac{2444}{3} + \frac{1768}{3}\nu\right)\widetilde{\sigma}_+^{(2)} - \frac{884}{3}\delta\,\widetilde{\sigma}_-^{(2)} - 130\widetilde{\mu}_+^{(3)}\right]x^7\Bigg\}
\end{aligned}$$

[HFB19]

Results: phasing to 2.5PN for circular orbits

$$\phi_{\text{pp}} = -\frac{1}{32\nu x^{5/2}}\left\{1 + \left(\frac{3715}{1008} + \frac{55}{12}\nu\right) x - 10\pi x^{3/2} + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2\right) x^2 + \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi x^{5/2}\ln\left(\frac{x}{x_0}\right)\right\}$$

$$\begin{aligned}\phi_{\text{tidal}} = -\frac{3x^{5/2}}{16\nu^2}\Bigg\{&(1+22\nu)\widetilde{\mu}_+^{(2)} + \delta\,\widetilde{\mu}_-^{(2)}\\
&+ \left[\widetilde{\mu}_+^{(2)}\left(\frac{195}{56} + \frac{1595}{14}\nu + \frac{325}{42}\nu^2\right) + \delta\,\widetilde{\mu}_-^{(2)}\left(\frac{195}{56} + \frac{4415}{168}\nu\right) + \widetilde{\sigma}_+^{(2)}\left(-\frac{5}{63} + \frac{3460}{21}\nu\right) - \frac{5}{63}\delta\,\widetilde{\sigma}_-^{(2)}\right] x\\
&- \frac{5\pi}{2}\left[(1+22\nu)\widetilde{\mu}_+^{(2)} + \delta\,\widetilde{\mu}_-^{(2)}\right] x^{3/2}\\
&+ \Bigg[\widetilde{\mu}_+^{(2)}\left(\frac{136190135}{9144576} + \frac{978554825}{1524096}\nu - \frac{281935}{2016}\nu^2 + 5\nu^3\right)\\
&\quad + \delta\,\widetilde{\mu}_-^{(2)}\left(\frac{136190135}{9144576} + \frac{213905}{864}\nu + \frac{1585}{432}\nu^2\right)\\
&\quad + \widetilde{\sigma}_+^{(2)}\left(-\frac{745}{1512} + \frac{1933490}{1701}\nu - \frac{3770}{27}\nu^2\right) + \delta\,\widetilde{\sigma}_-^{(2)}\left(-\frac{745}{1512} + \frac{19355}{81}\nu\right) + \frac{1000}{9}\nu\widetilde{\mu}_+^{(3)}\Bigg] x^2\\
&+ \pi\Bigg[\widetilde{\mu}_+^{(2)}\left(-\frac{397}{32} - \frac{5343}{16}\nu + \frac{1315}{12}\nu^2\right) + \delta\,\widetilde{\mu}_-^{(2)}\left(-\frac{397}{32} - \frac{6721}{96}\nu\right)\\
&\quad + \widetilde{\sigma}_+^{(2)}\left(\frac{1}{3} - \frac{4156}{9}\nu\right) + \frac{\delta}{3}\widetilde{\sigma}_-^{(2)}\Bigg] x^{5/2}\Bigg\}\end{aligned}$$

$\hookrightarrow$ Also computed in the stationary phase approximation [HFB20a]

Results: amplitude modes to 2.5PN for circular orbits

$$h_{\ell m} = \frac{8GM\nu x}{R\,c^2}\sqrt{\frac{\pi}{5}}\left(\hat{H}^{\text{pp}}_{\ell m} + \frac{x^5}{\nu}\hat{H}^{\text{tidal}}_{\ell m}\right)e^{-im\psi}$$

$$\begin{aligned}
\hat{H}^{\text{pp}}_{22} =& 1 + x\left(-\frac{107}{42}+\frac{55\nu}{42}\right) + 2\pi x^{3/2} + x^2\left(-\frac{2173}{1512}-\frac{1069\nu}{216}+\frac{2047\nu^2}{1512}\right)\\
&+ x^{5/2}\left(-\frac{107\pi}{21} - 24i\nu + \frac{34\pi\nu}{21}\right)\\
\hat{H}^{\text{tidal}}_{22} =& \widetilde{\mu}^{(2)}_{+}(3+12\nu) + 3\delta\,\widetilde{\mu}^{(2)}_{-} + \left[\widetilde{\mu}^{(2)}_{+}\left(\frac{9}{2}-20\nu+\frac{45}{7}\nu^2\right) + \delta\,\widetilde{\mu}^{(2)}_{-}\left(\frac{9}{2}+\frac{125}{7}\nu\right) + \frac{224}{3}\nu\,\widetilde{\sigma}^{(2)}_{+}\right]x\\
&+ 6\pi\left[\widetilde{\mu}^{(2)}_{+}(1+4\nu) + \delta\,\widetilde{\mu}^{(2)}_{-}\right]x^{3/2}\\
&+ \left[\widetilde{\mu}^{(2)}_{+}\left(\frac{1403}{56}-\frac{7211}{168}\nu-\frac{19367}{168}\nu^2-\frac{274}{21}\nu^3\right) + \delta\,\widetilde{\mu}^{(2)}_{-}\left(\frac{1403}{56}+\frac{1559}{56}\nu+\frac{103}{24}\nu^2\right)\right.\\
&\quad\left. + \widetilde{\sigma}^{(2)}_{+}\left(\frac{11132}{63}\nu - \frac{6536}{63}\nu^2\right) + \frac{8084}{63}\delta\,\nu\,\widetilde{\sigma}^{(2)}_{-} + 80\nu\,\widetilde{\mu}^{(3)}_{+}\right]x^2\\
&+ \left[\widetilde{\mu}^{(2)}_{+}\left(\frac{i}{5}(64-108\nu-8640\nu^2) + \frac{\pi}{7}(63-301\nu+132\nu^2)\right)\right.\\
&\quad\left. + \delta\,\widetilde{\mu}^{(2)}_{-}\left(\frac{i}{5}(64+20\nu)+\frac{\pi}{7}(63+229\nu)\right) + \frac{448}{3}\pi\,\nu\,\widetilde{\sigma}^{(2)}_{+}\right]x^{5/2}
\end{aligned}$$

- $\hookrightarrow$ Much harder to compute than the phase
- $\hookrightarrow$ Computed each mode for $2 \le \ell \le 7$ and $0 \le |m| \le \ell$
- $\hookrightarrow$ Full waveform completed to 2.5PN beyond leading order in [DHB24]

Implementation in waveform models

↪ The phasing is implemented in all waveform models for tidal effects

↪ Amplitude factorized in a convenient form for EOB formalism

$$h_{\ell m}^{\mathrm{EOB}} = h_{\ell m}^{\mathrm{N}} S_{\mathrm{eff}} T_{\ell m} f_{\ell m} e^{i\delta_{\ell m}}$$

- **EOB**
  ↪ Implemented in `TEOBResumS` [Gamba et al. 2022]
  ↪ Implemented in `SEOBNRv5THM` [Haberland et al. 2025]

- **Phenomenological**
  ↪ Implemented in `NRTidalv3` [Abac et al. 2023]
  ↪ `IMRPhenomXPHM_NSBH`, see Felip's talk [To be published]

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Conclusion

Summary:

- We used an effective action describing adiabatic tides
- We applied the PN-MPM formalism to this action
  $\hookrightarrow$ Computation of dynamics and conserved quantities to 2.5PN
  $\hookrightarrow$ Computation of the radiated fluxes and the phase to 2.5PN
  $\hookrightarrow$ Computation of the amplitude modes to 2.5PN
  $\hookrightarrow$ The amplitude is much harder to compute than the phase
- Implemented in all waveform models dedicated to matter effects

Perspectives:

- Relax the quasi-ciruclar orbits approximation
  $\hookrightarrow$ Compute the full waveform for eccentric orbits
- Relax the adiabatic approximation
  $\hookrightarrow$ Dynamical tides can play an important qualitative role in BNS

Acknowledgments

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