Efficient methods for wave-optics lensing

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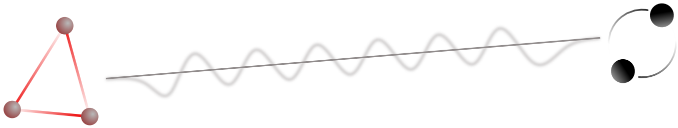
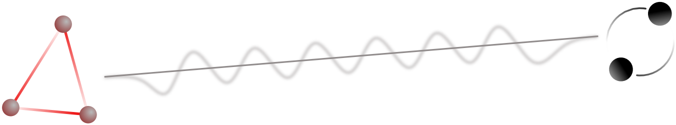
GW astronomy



- Hard to detect but very clean (good models)
- Propagate through ‘opaque’ regions
- Poor source localization
- Direct measurement of distance, redshift unknown
- Measure of amplitude rather than intensity (amplitude^2)

GW lensing



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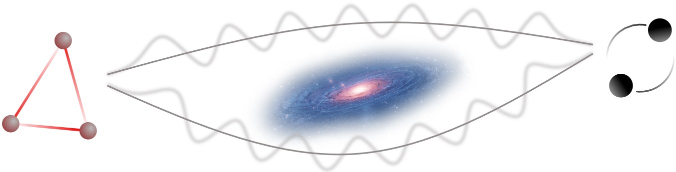
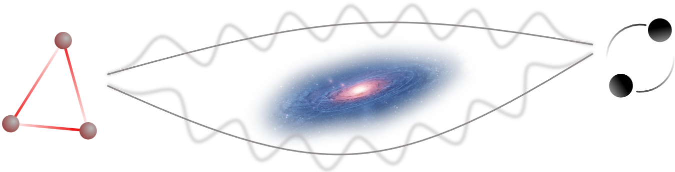
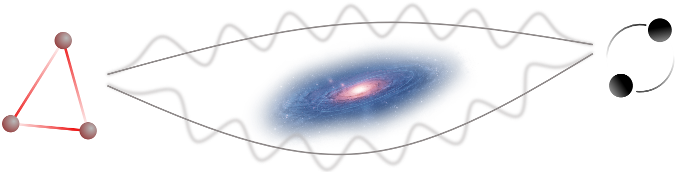




Why should we care?



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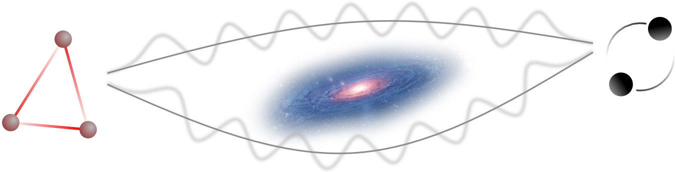

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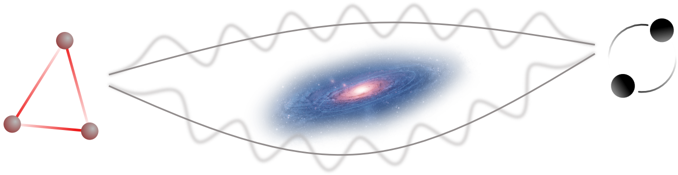
- We will see lensed events
- Undetected lensing can bias the parameters of the source



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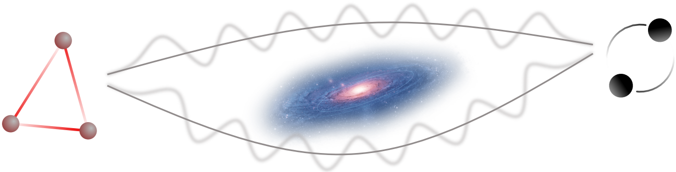
- GWs complement electromagnetic lensing



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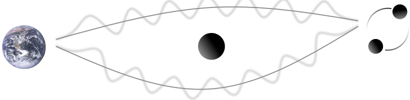
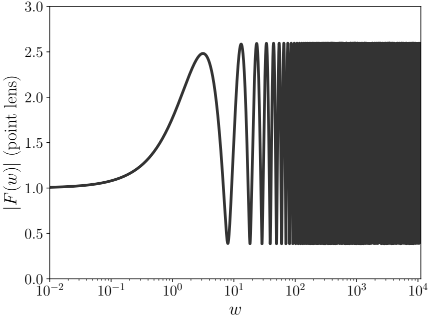
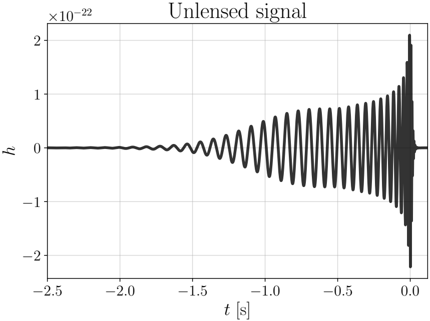
Lensing effects are encapsulated in the amplification factor:

$$h_L(w) = F(w)h_0(w)$$

Strong lensing with LVK/ET







$30 + 30\ M_\odot$ BBH at $z = 1$

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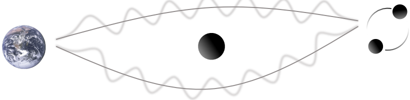

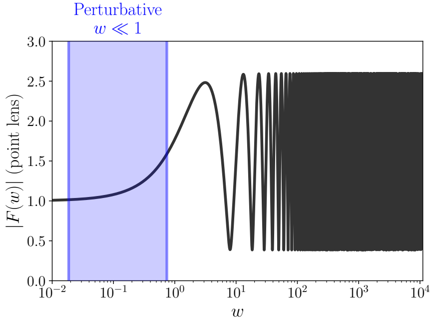

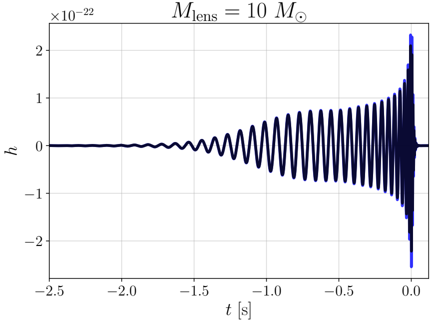

$$h_L(w) = F(w) h_0(w)$$

$$F(w) = 1 + \mathcal{O}(w)$$

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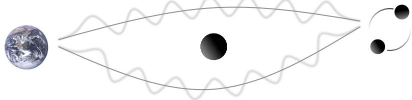

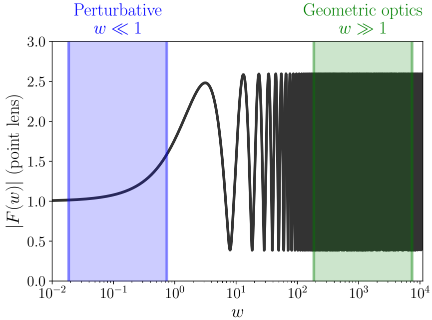

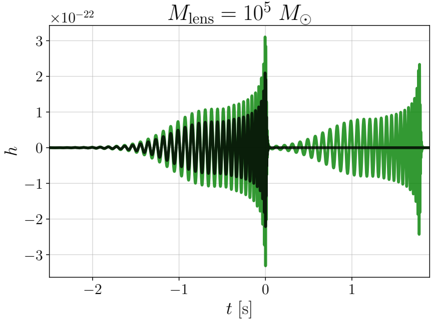

$$h_L(w) = F(w) h_0(w)$$

$$F(w) = \sum_I \sqrt{\mu}_I e^{iw\Delta\tau_I + i\pi n_I} + \mathcal{O}\left(w^{-1}\right)$$

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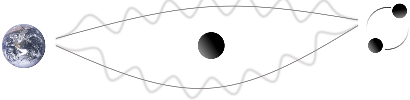
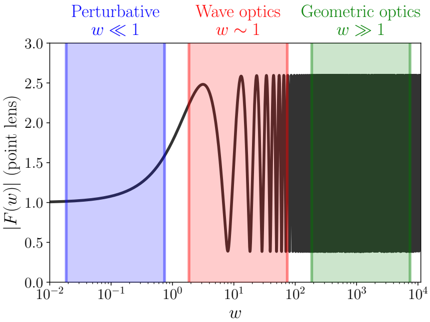
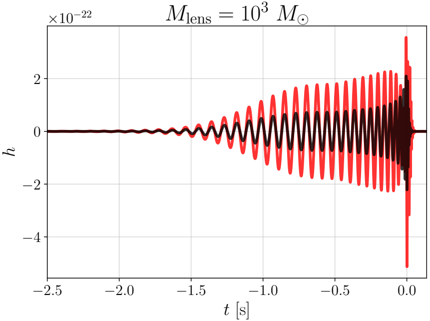
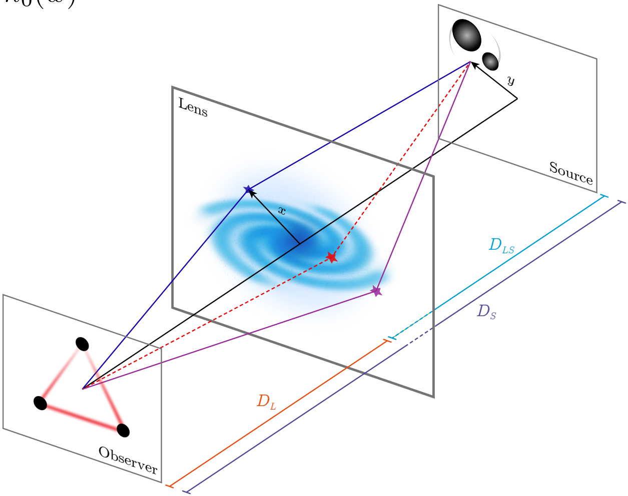






$$h_L(w) = F(w) h_0(w)$$
$$F(w) = ?$$

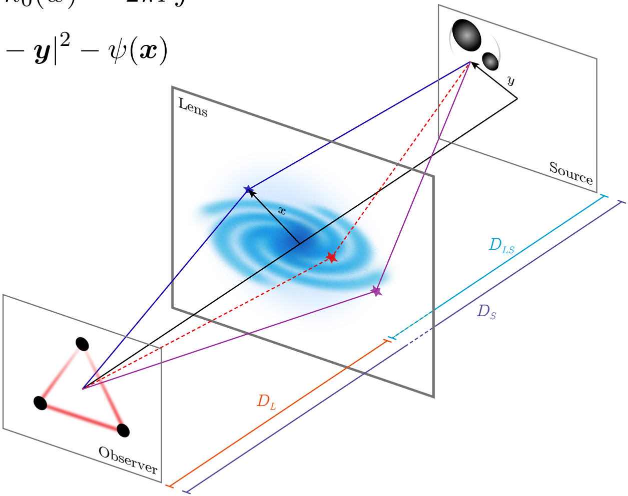
$$F(w) = \frac{h_L(w)}{h_0(w)}$$



[Tambalo, Zumalacárregui, Dai, Ho-Yeuk Cheung, 2212.11960]

$$F(w) = \frac{h_L(w)}{h_0(w)} = \frac{w}{2\pi \mathrm{i}} \int \mathrm{d}^2 x \exp(\mathrm{i} w \phi)$$

$$\phi = \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}|^2 - \psi(\boldsymbol{x})$$



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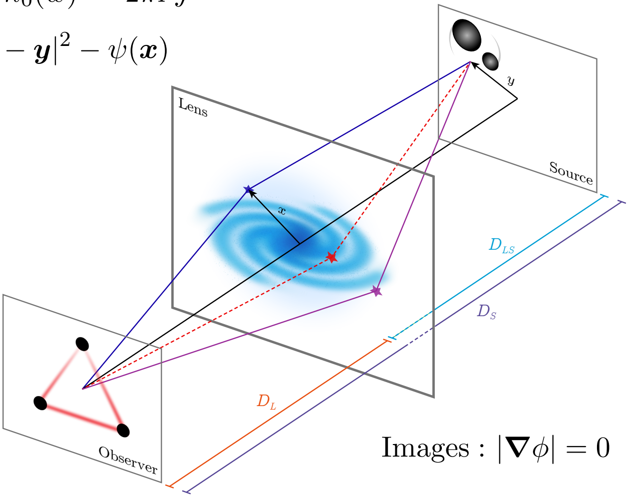

Images : $|\boldsymbol{\nabla}\phi| = 0$

[Tambalo, Zumalacárregui, Dai, Ho-Yeuk Cheung, 2212.11960]

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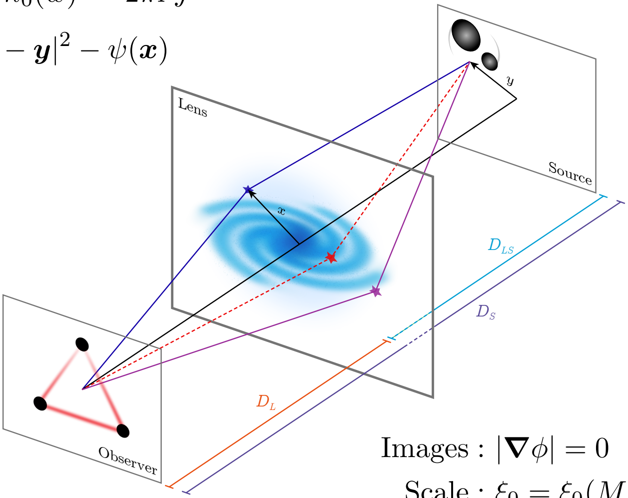

$$\text{Images}: |\boldsymbol{\nabla}\phi| = 0$$

$$\text{Scale}: \xi_0 = \xi_0(M, z_L, z_S, \dots)$$

$$y = y_{\text{phys}}/\xi_0$$

$$w = \xi_0 f_{\text{phys}}$$

[Tambalo, Zumalacárregui, Dai, Ho-Yeuk Cheung, 2212.11960]

Amplification factor:

$$F(w) = \frac{h_L(w)}{h_0(w)} = \frac{w}{2\pi \mathrm{i}} \int \mathrm{d}^2 x \exp(\mathrm{i} w \phi)$$

Simplest approach: geometric optics

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Compute the critical points $|\boldsymbol{\nabla}\phi| = 0$ and use the stationary phase approximation

$$F(w) \simeq \sum_i \sqrt{\mu_i}\, \mathrm{e}^{\mathrm{i} w \tau_i + \mathrm{i}\pi n_i} + \mathcal{O}(1/w)$$

This is the *geometric optics* (GO) result.

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- ✓ Very simple and fast
- ✗ Misses completely the wave-optics effects ($w \sim 1$)

Our approach: time-domain

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$$F(w) = \frac{w}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\tau \, \mathrm{e}^{\mathrm{i} w \tau} I(\tau)$$

$$I(\tau) = \int \mathrm{d}^2 x \, \delta\Big(\phi(\boldsymbol{x}) - \tau\Big)$$

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Grid method

- J.M. Diego+ 2019
- X. Shan+ 2023, 2024, `Microlensing_Wave_Effect`
- S.M.C. Yeung+ 2024, `wolensing`

Contour method

- Ulmer, Goodman 1994
- Tambalo+ 2022, 2023
- M.H.-Y. Cheung 2023, `glworia`

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✗ Hard to implement in full generality

✓ Already implemented → `GLoW`

Contour method

- Ulmer, Goodman 1994
- Tambalo+ 2022, 2023
- M.H.-Y. Cheung 2023, **`glworia`**

Contour method: Step by step

(1) Choose a lens, $\psi(\boldsymbol{x})$

(2) Choose a source position, $\boldsymbol{y}$

(3) Find critical points, $|\boldsymbol{\nabla}\phi| = 0$

(4) Compute the time-domain amplification factor, $I(\tau)$

(5) Transform to the frequency-domain, $F(w)$

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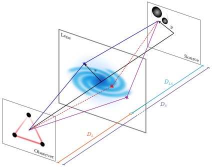
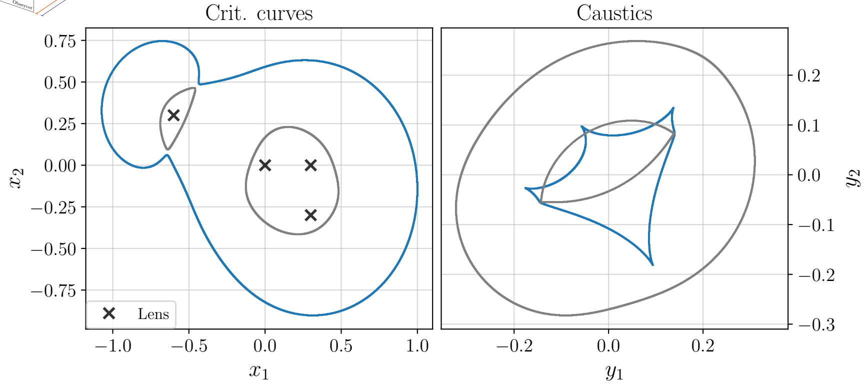
(6) Choose a source, $h_0(f)$

(7) Compute the lensed waveform, $h_L(f) = F(f)h_0(f)$



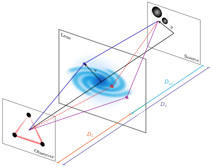
(1) Lens choice: 4 CISs

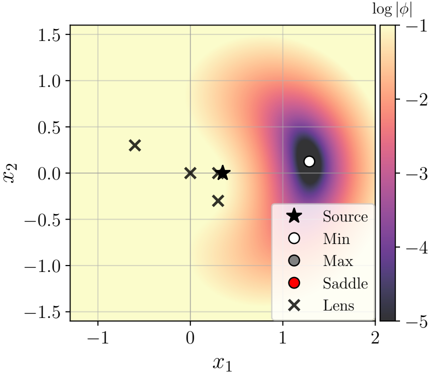


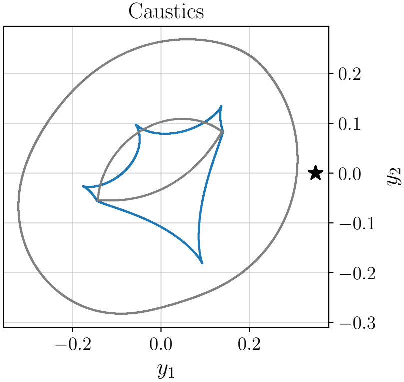
$$\psi(\boldsymbol{x}) = \sum_i \psi_{\text{CIS}}(\boldsymbol{x} - \boldsymbol{x}_0^i) \ ,$$

CIS : Cored Isothermal Sphere

(2), (3) Find images







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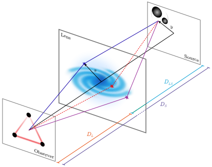

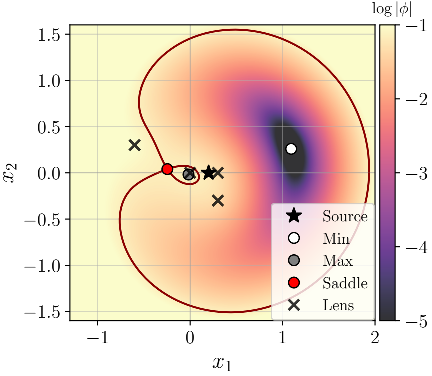

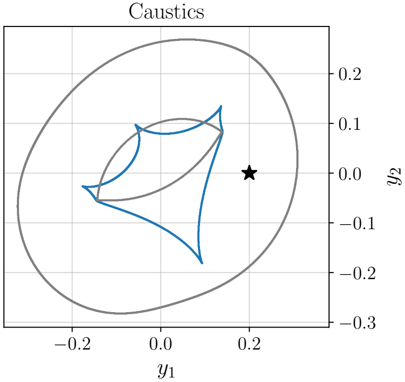

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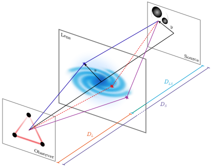

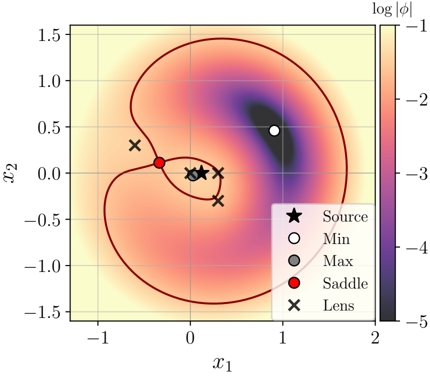

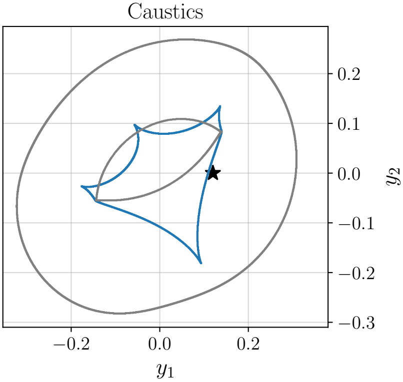

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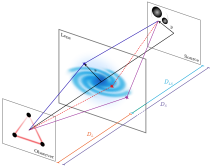

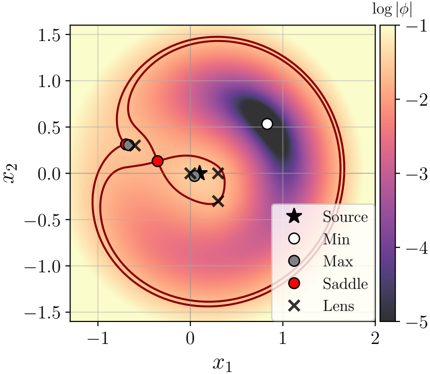

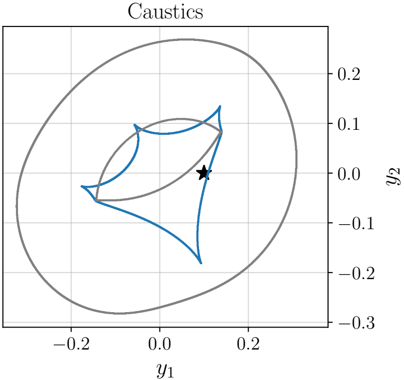

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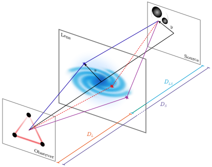

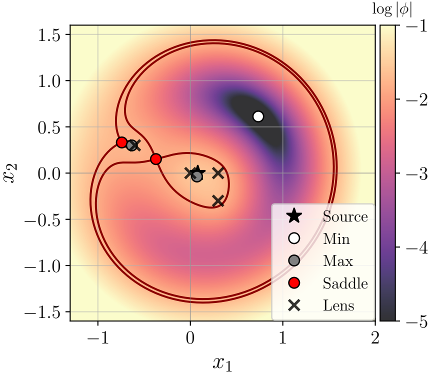

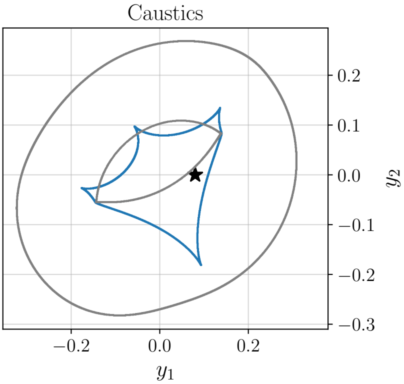

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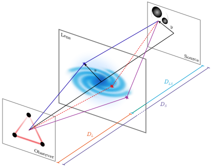

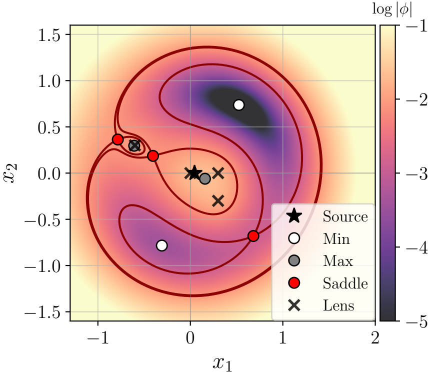

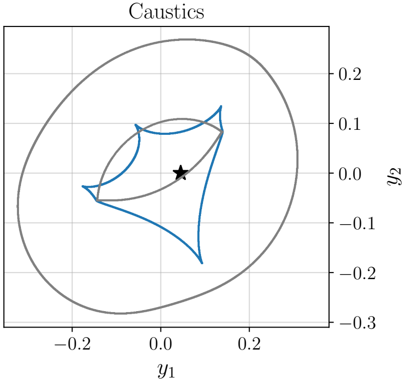

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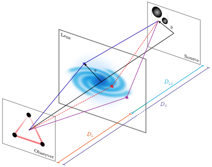

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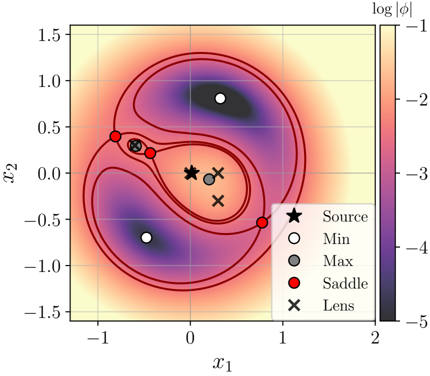

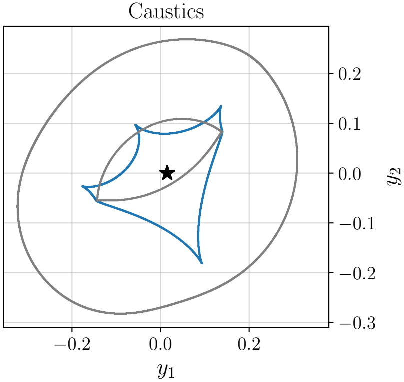

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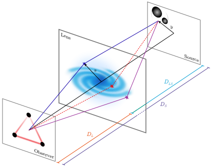
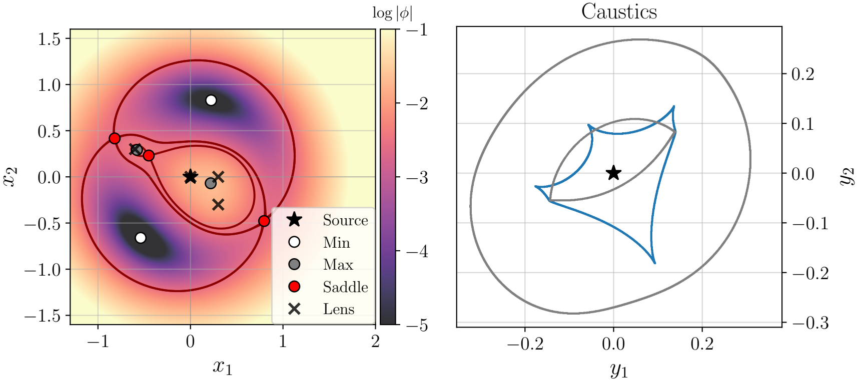
(2), (3) Find images





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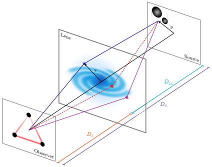

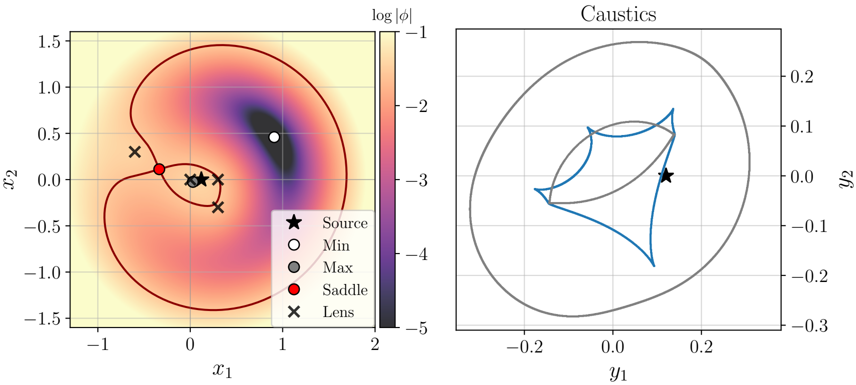

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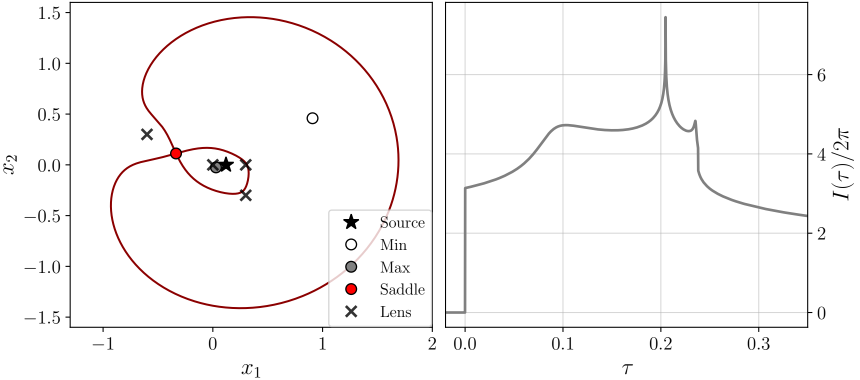

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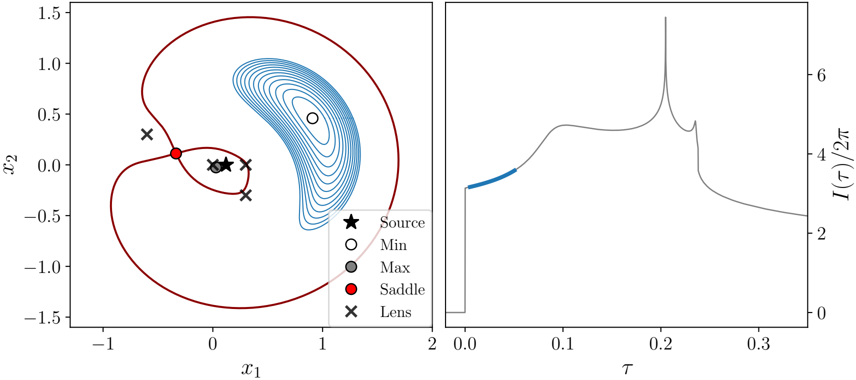

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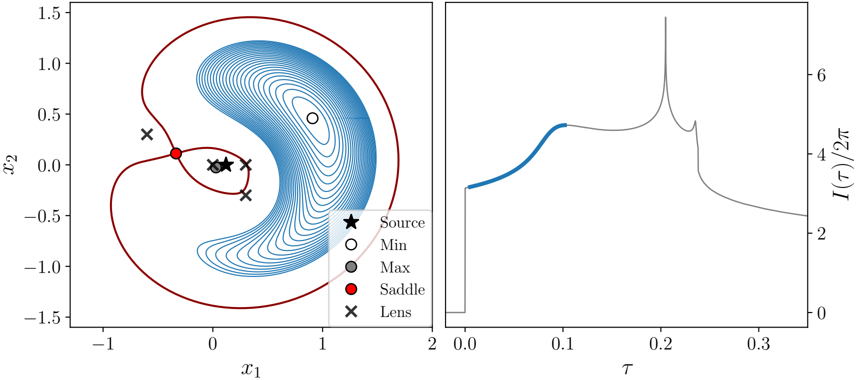

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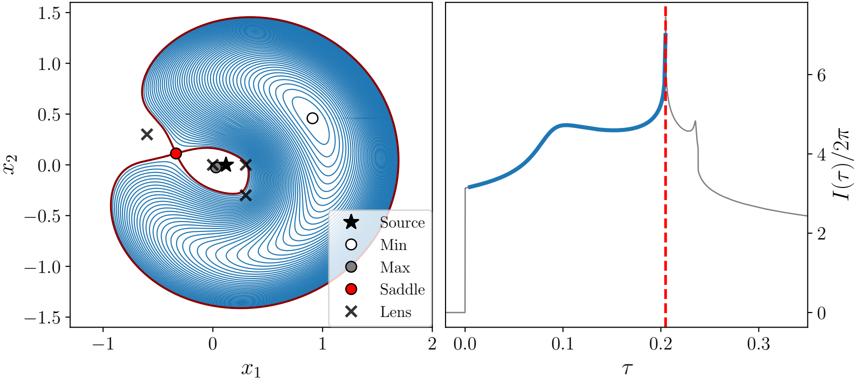

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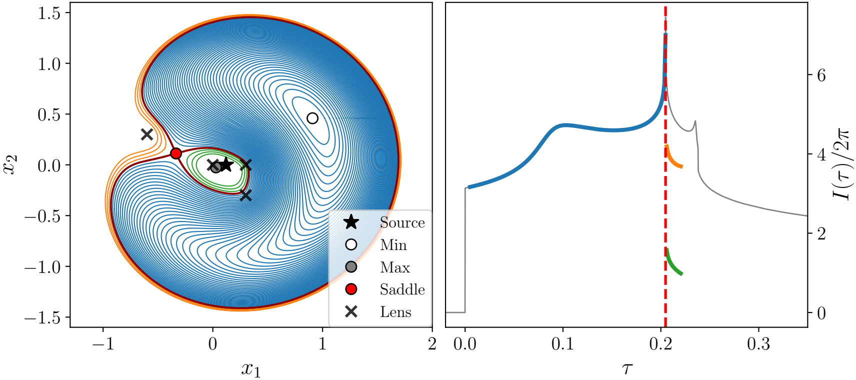

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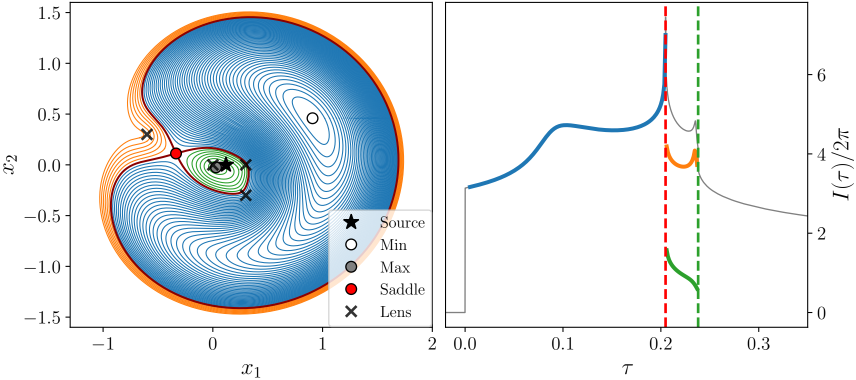

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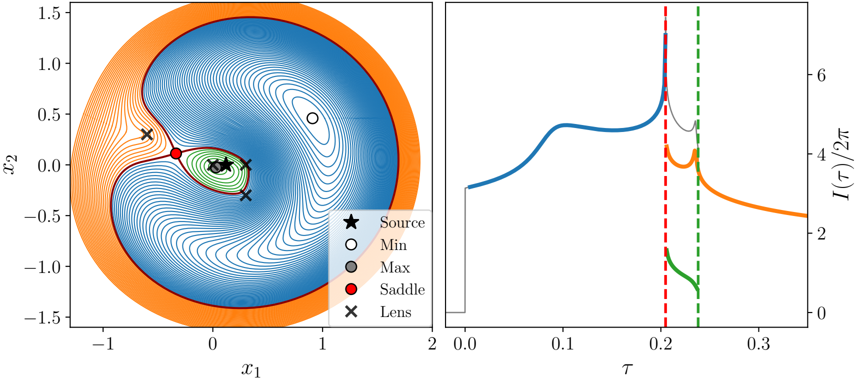

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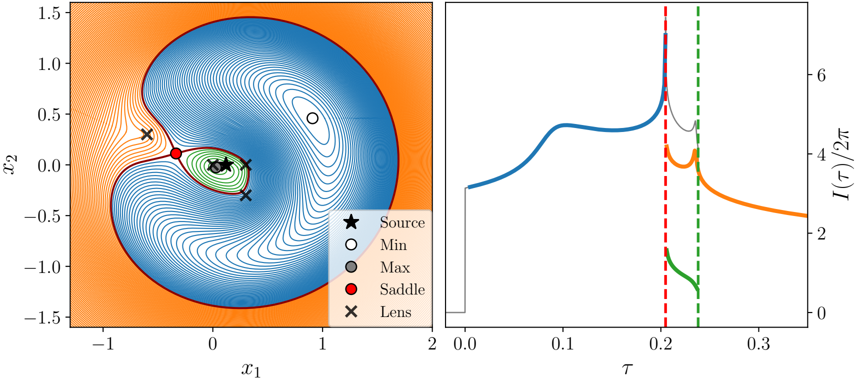

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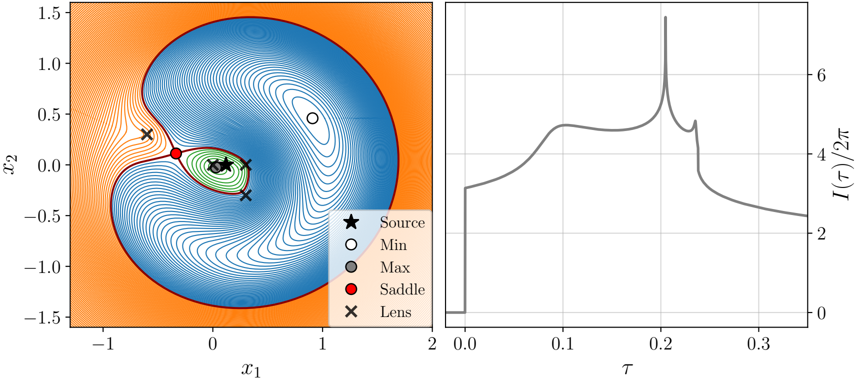

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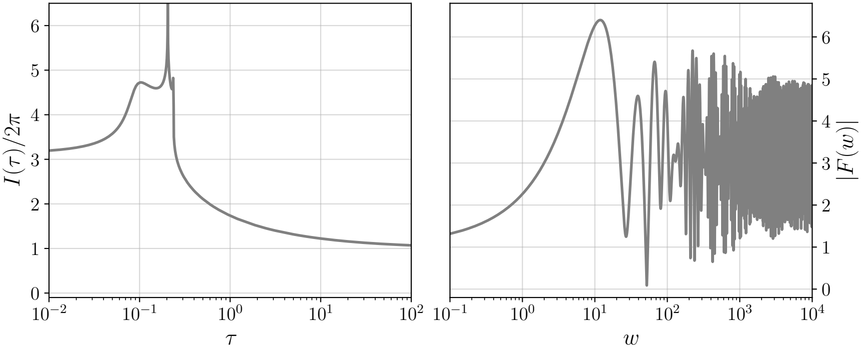
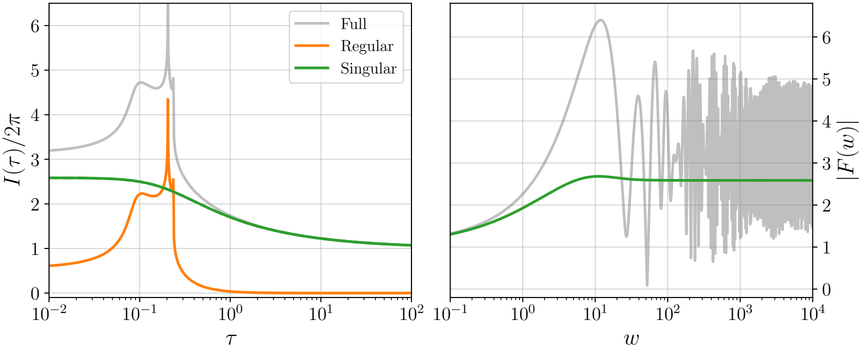
$$F(w) = \frac{w}{2\pi\mathrm{i}}\int \mathrm{d}\tau\ \mathrm{e}^{\mathrm{i}w\tau} I(\tau)$$



(5) Regularization

$$F(w) = \frac{w}{2\pi \mathrm{i}} \int \mathrm{d}\tau \; \mathrm{e}^{\mathrm{i}w\tau} I(\tau)$$

Regular = **Full** − **Singular**



$$\text{Tail} \sim 1/\tau^{\alpha} \quad + \quad \text{Minimum} \sim \sqrt{\mu_0}\,\Theta(\tau - \tau_0)$$

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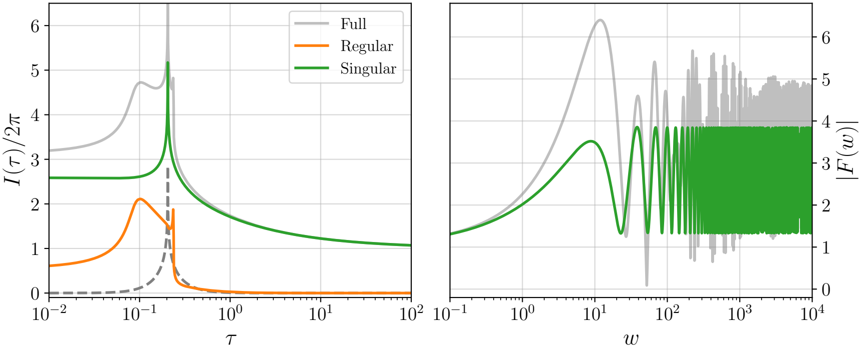

Saddle $\sim \sqrt{\mu_0}\, \log|\tau - \tau_0|$

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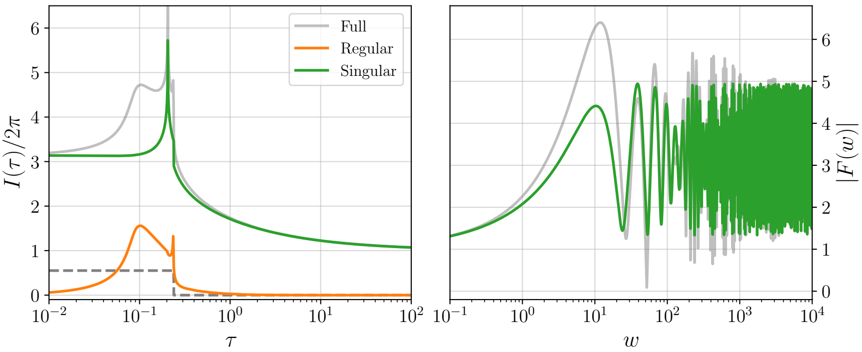
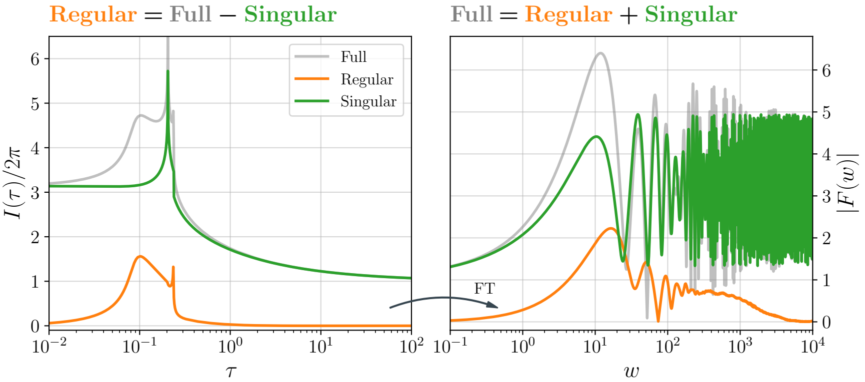


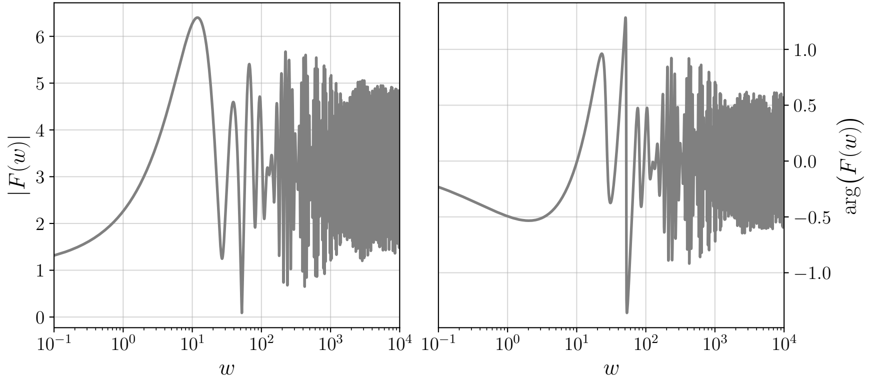
Maximum $\sim \sqrt{\mu_0}\,\Theta(\tau_0 - \tau)$

$$F(w) = \frac{w}{2\pi\mathrm{i}} \int \mathrm{d}\tau \; \mathrm{e}^{\mathrm{i}w\tau} I(\tau)$$



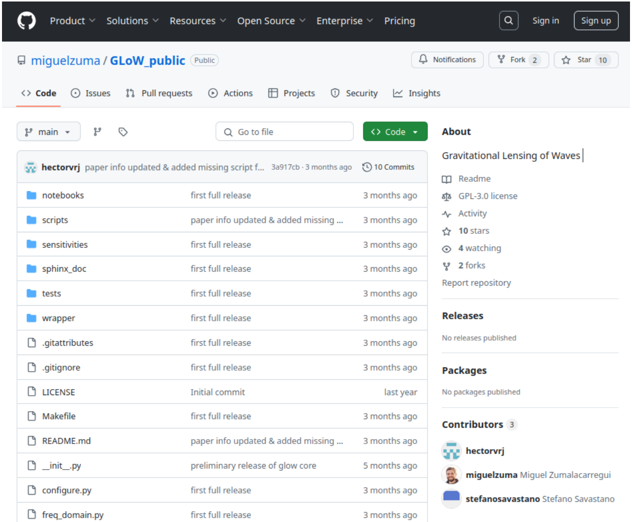
Analytic F_{sing} + Fourier Transform of I_{reg} = $F(w)$

(6) Compute lensed waveform



Lensed waveform: $h_L(w) = F(w)h_0(w)$

Gravitational Lensing of Waves (GLoW)



`miguelzuma/GLoW_public`

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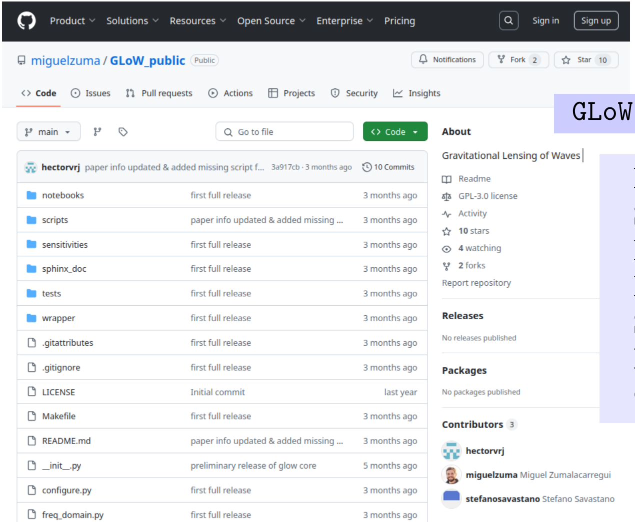

GLoW paper: 2409.04606

HVR
S. Savastano
M. Zumalacárregui
L. Choi
S. Goyal
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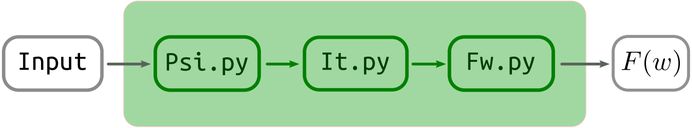
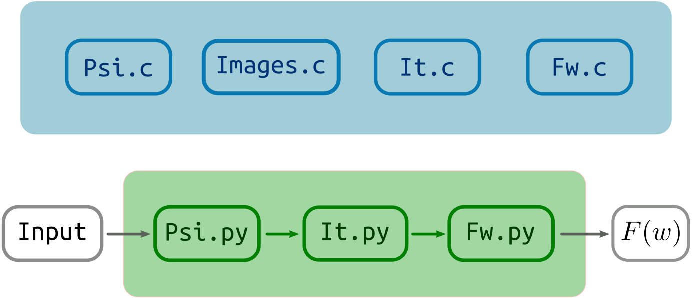
miguelzuma/GLoW_public



- *General*: non-symmetric lenses, strong and weak lensing
- *Flexible*: can be used as a Python module (also as a C lib)
- *Fast*: typical run $\sim \mathcal{O}(10 \text{ ms})$



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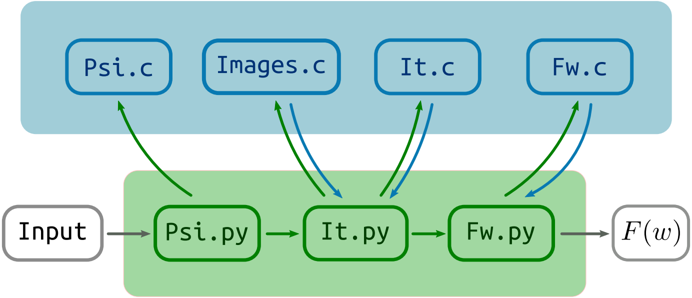
GLoW highlights



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```
[1]: import numpy as np
     import matplotlib.pyplot as plt

     from glow.lenses import Psi_offcenterCIS, CombinedLens
     from glow import time_domain_c, freq_domain_c
```

```
[2]: # create the lens
     xs = [(0.3, 0), (-0.6, 0.3), (0.3, -0.3), (0, 0)]
     Psis = [Psi_offcenterCIS({'psi0':0.25, 'rc':0.05, 'xc1':x1, 'xc2':x2}) for x1, x2 in xs]
     Psi = CombinedLens({'lenses':Psis})

     # compute the amplification factor
     It = time_domain_c.It_MultiContour_C(Psi, y=0.12)
     Fw = freq_domain_c.Fw_FFT_C(It, p_prec={'wmin':1e-1, 'wmax':1e4})
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```
[3]: ws = np.geomspace(1e-1, 1e4, 2000)

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GLoW sample

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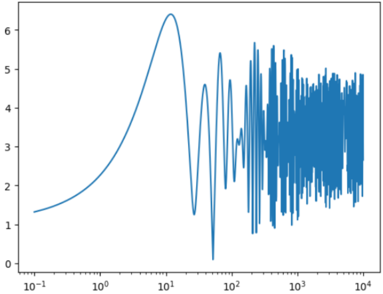

Summary of `GLoW`'s features

- Computation of the diffraction integral in the time and frequency domains, for generic (analytic) lenses and arbitrary frequencies
- New optimized method for symmetric lenses
- New regularization scheme
- New expressions for the SIS, with an analytical $I(\tau)$ and semi-analytical $F(w)$
- Fast implementation of the analytic $F(w)$ for the point lens
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Approach 2: brute force

Amplification factor:

$$F(w) = \frac{h_L(w)}{h_0(w)} = \frac{w}{2\pi \mathrm{i}} \int \mathrm{d}^2 x \exp(\mathrm{i} w \phi)$$

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$$F(w) = -\mathrm{i} w \, \mathrm{e}^{\mathrm{i} w y^2/2} \int_0^\infty \mathrm{d}x \, x J_0(w y x) \mathrm{e}^{\mathrm{i} w (x^2/2 - \psi(x))}$$

Then integrate this expression with arbitrary-precision software (Mathematica, ARB). E.g. [Wright, Hendry, 2022, GRAVELAMPS]

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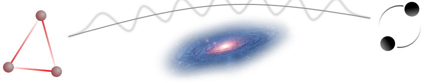
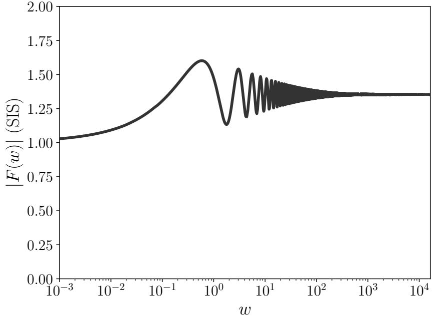
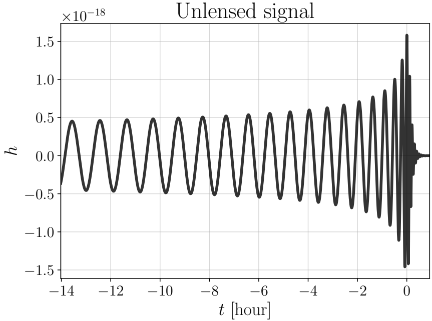
Then integrate this expression with arbitrary-precision software (Mathematica, ARB). E.g. [Wright, Hendry, 2022, GRAVELAMPS]

- ✓ Simple to implement
- ✗ Restricted to symmetric lenses
- ✗ Slow (too slow for parameter estimation)

(2) Weak lensing with LISA







$10^6 + 10^6\ M_\odot$ BBH at $z = 5$

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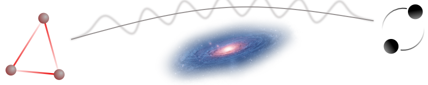

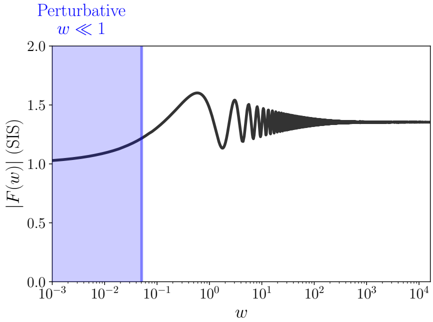

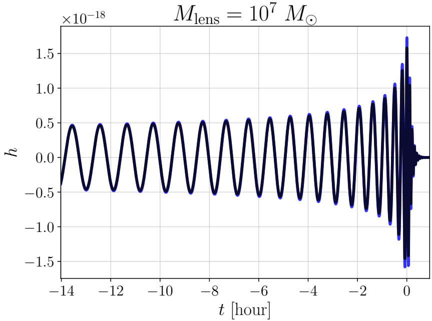

$$h_L(w) = F(w)h_0(w)$$

$$F(w) = 1 + \mathcal{O}(w)$$

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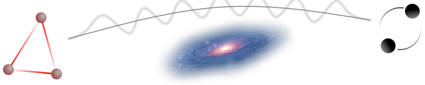
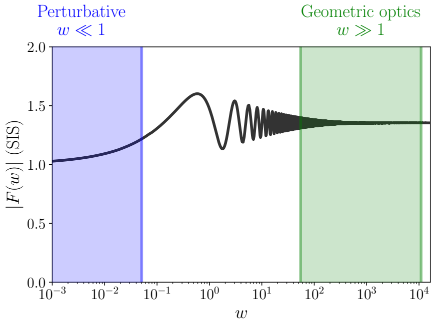
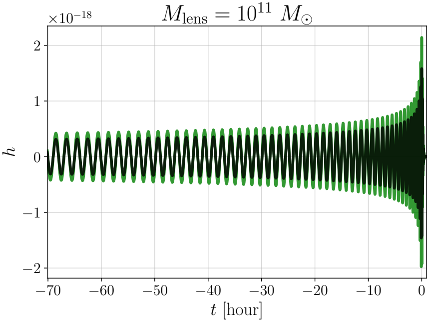


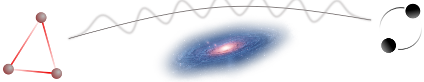


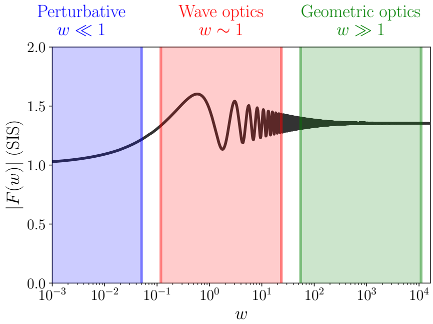


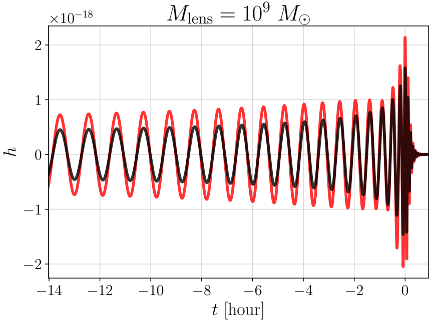
$$h_L(w) = F(w)h_0(w)$$

$$F(w) = \sqrt{\mu}\,\mathrm{e}^{\mathrm{i}w\Delta\tau} + \mathcal{O}\left(w^{-1}\right)$$

(2) Weak lensing with LISA







$$h_L(w) = F(w) h_0(w)$$

$$F(w) = ?$$

Time-domain amplification factor

$$F(w) = \frac{w}{2\pi \mathrm{i}} \int \mathrm{d}\tau \, \mathrm{e}^{\mathrm{i}w\tau} I(\tau)$$

$$h_L(w) = F(w) h_0(w)$$

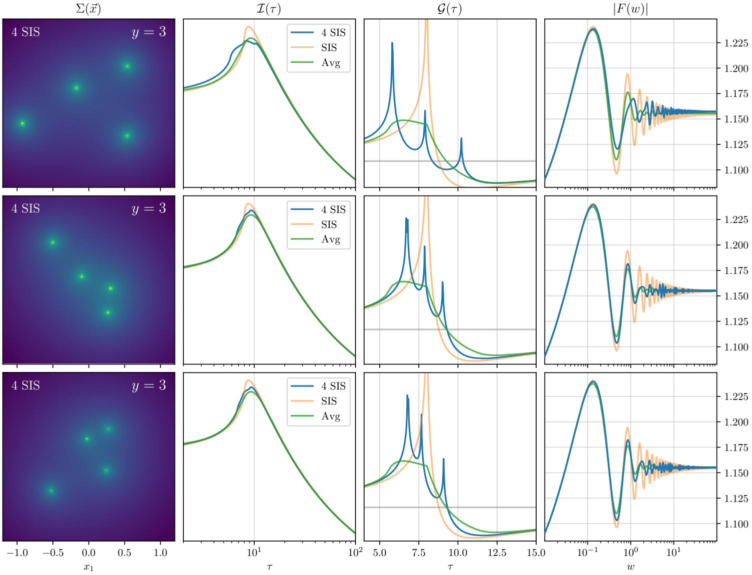
$$I(\tau) = \int \mathrm{d}^2 x \; \delta\left(\phi(\boldsymbol{x} - \tau)\right)$$

$$G(\tau) = \frac{1}{2\pi} \frac{\mathrm{d}I}{\mathrm{d}\tau}$$

$$h_L(\tau) = \int \mathrm{d}\tau' G(\tau' - \tau) h_0(\tau')$$

[Savastano, Tambalo, HVR, Zumalacárregui, 2306.05282]

Example: variying distribution of lenses



[Savastano, Tambalo, HVR, Zumalacárregui, 2306.05282]

Example: variying number of lenses